Geometric Methods for Curve Classification: *Mathematical Handwriting Recognition*

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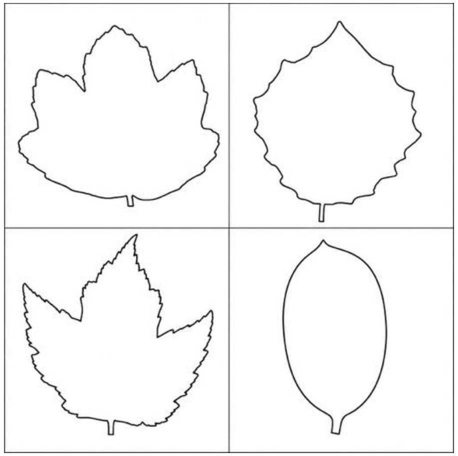
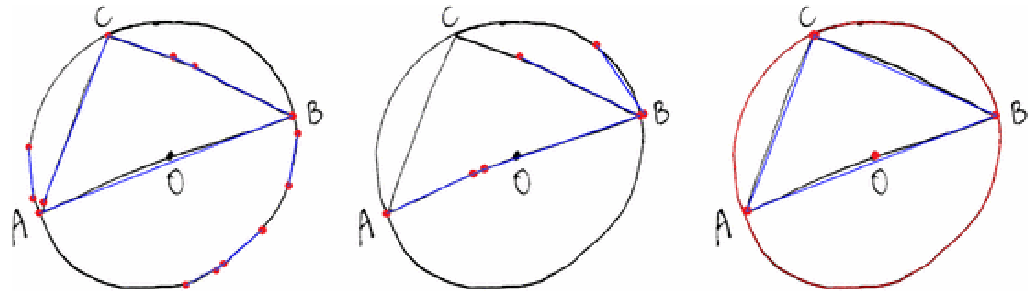
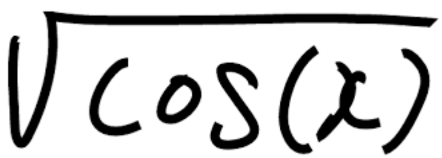
Curve Classification is Everywhere



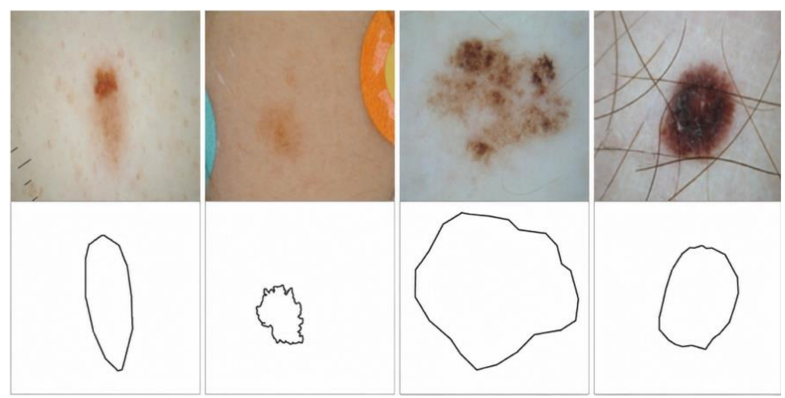
https://github.com/sunsided/kaggle-leaf-classification



https://link.springer.com/article/10.1007/s10618-017-0494-1



https://link.springer.com/chapter/10.1007/978-3-031-41679-8_33



https://challenge.isic-archive.com/landing/2018/45/

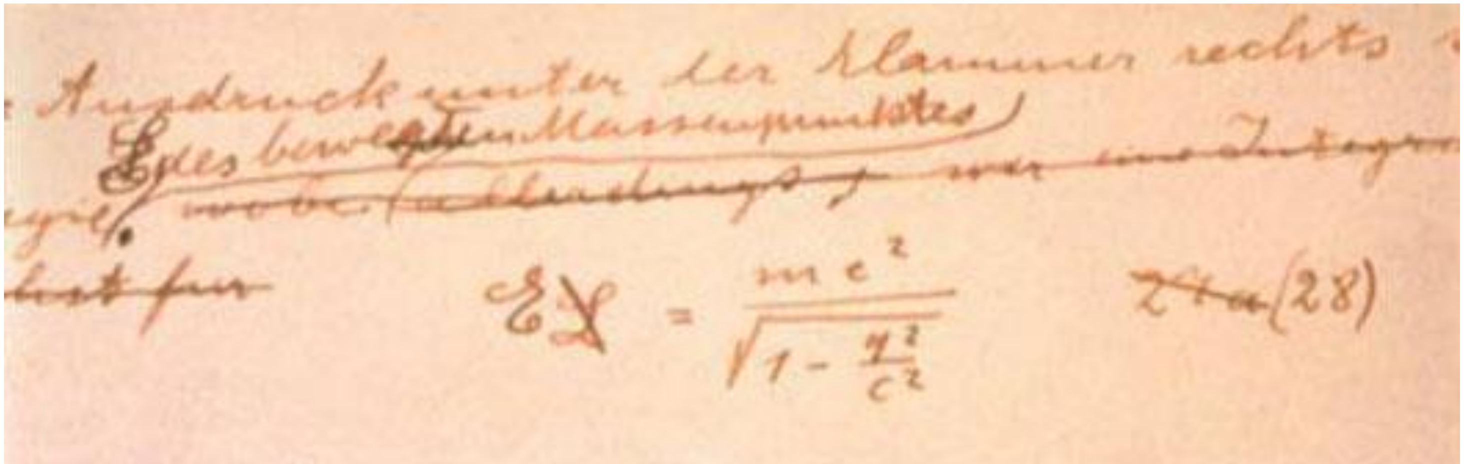
The Common Problem

- Given a sampled curve γ and a set of labelled model curves Σ, classify γ by shape.

- Examples:
  * leaf margin → species
  * lesion border → diagnostic class
  * ink trace → mathematical symbol

- We want a representation that is *compact*, geometric, *fast*, and *trainable from few examples*.

Pen-Based Math



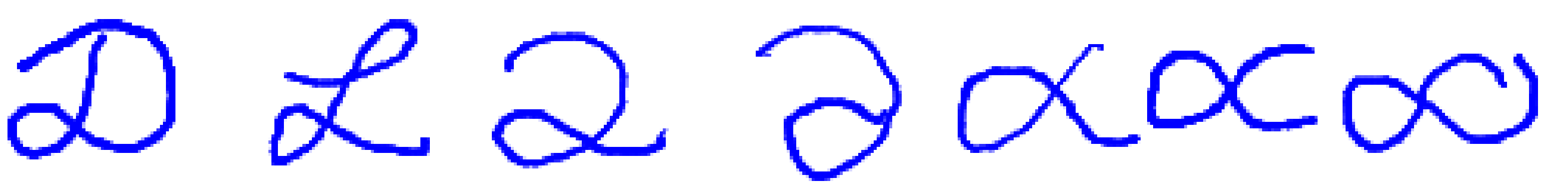
- Input for CAS and document processing.
- 2D editing.
- Computer-assisted collaboration.

Why not just use Deep Neural Networks?

- Deep neural networks are excellent when training data is abundant.
- Here the regime is different:
  * few examples per class
  * new users and idiosyncratic symbols
  * need fast interactive response
  * want meaningful distances between shapes
  * want algebraic operations on curves
- What geometry does the data already have?

Pen-Based Math

- Different than natural language recognition:
  - 2-D layout is a combination of writing and drawing.
  - Many similar few-stroke characters.
  - Many alphabets, used idiosyncratically.
  - Many symbols, each person uses a subset.
  - No fixed dictionary for disambiguation.



Character Recognition

- A story about a UI proposal to Maplesoft
- A story about three statisticians
- Will concentrate on character recognition
- Several projects ignoring this problem

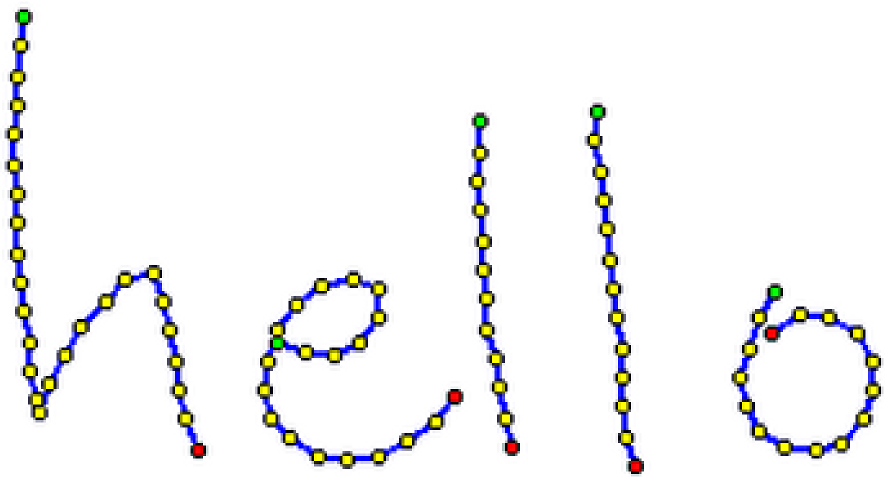
Digital Ink

- Location, time information, sometimes also pressure and angles.
- Capture online pen strokes, *not* images.

- Suitable for
  - **Recognition** algorithms
  - **Semantic** grouping
  - **Annotation**
  - **Manipulation**: search, transformation, archival.

- Problem: Multiple vendor-specific formats.

Digital Ink Formats



- Collected by surface digitizer or camera
- Sequence of (x, y) points sampled at some known frequency
- Possibly other info (angles, pressure, etc)
- Grouping into traces, letters, words + labelling

W3C Recommendation

Ink Markup Language (InkML)

W3C Recommendation 20 September 2011

This version:
: http://www.w3.org/TR/2011/REC-InkML-20110920/

Latest version:
: http://www.w3.org/TR/InkML

Previous version:
: http://www.w3.org/TR/2011/PR-InkML-20110510/

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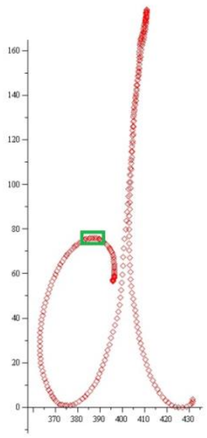
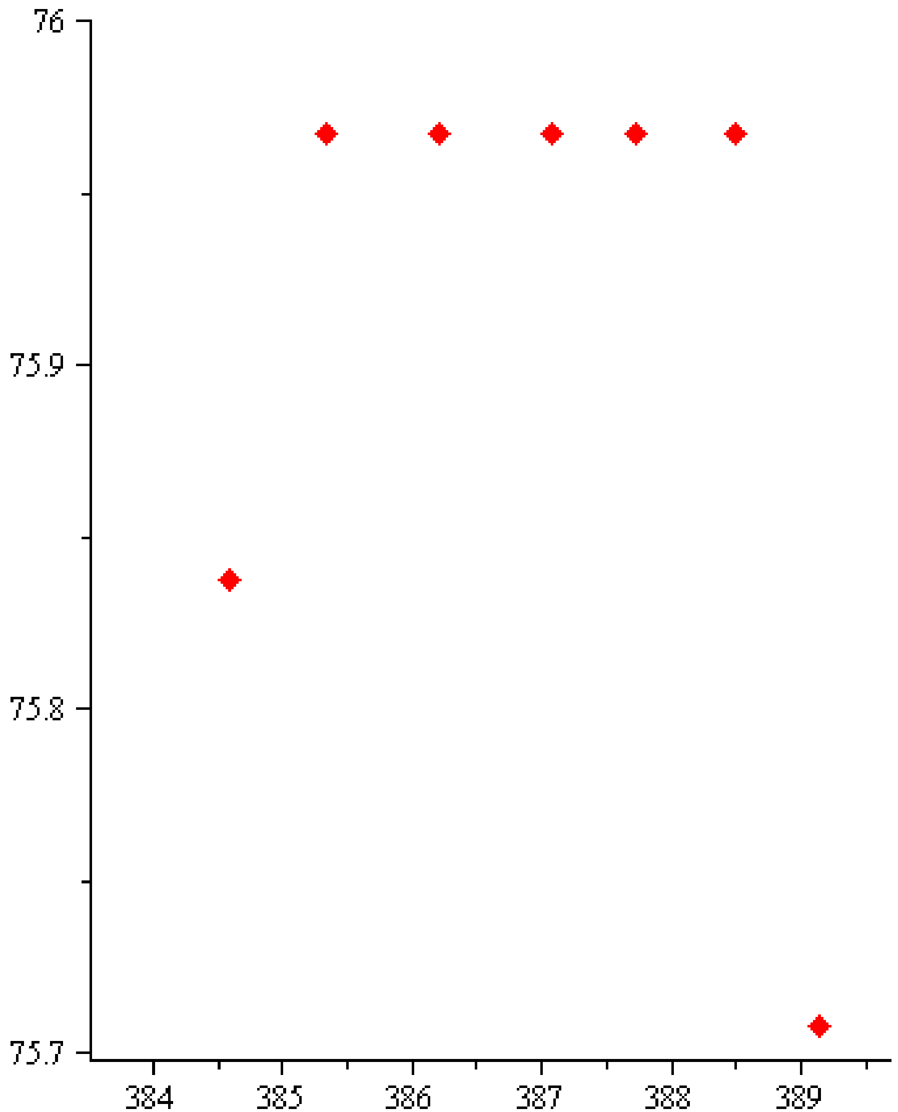
What the Computer Sees





Usual Character Reco. Methods

- Smooth and re-sample data *THEN*
- Match against *N* models by sequence alignment
  OR
- Identify "features", such as
  - coordinate values of sample points, number of loops, cusps, writing direction at selected points, *etc*

  Use a classification method, such as
  - Nearest neighbour, Subspace projection, Cluster analysis, Support Vector Machine

  THEN
- Rank choices by consulting dictionary

Difficulties

- Having many similar characters (*e.g.* for math) means comparison against all possible symbol models is slow.
- Determining features from points
  - Requires many *ad hoc* parameters.
  - Replaces measured points with interpolations
  - It is not clear how many points to keep, and most methods depend on number of points
  - Device dependent
- What to do since there is no dictionary?
- New ideas are needed!

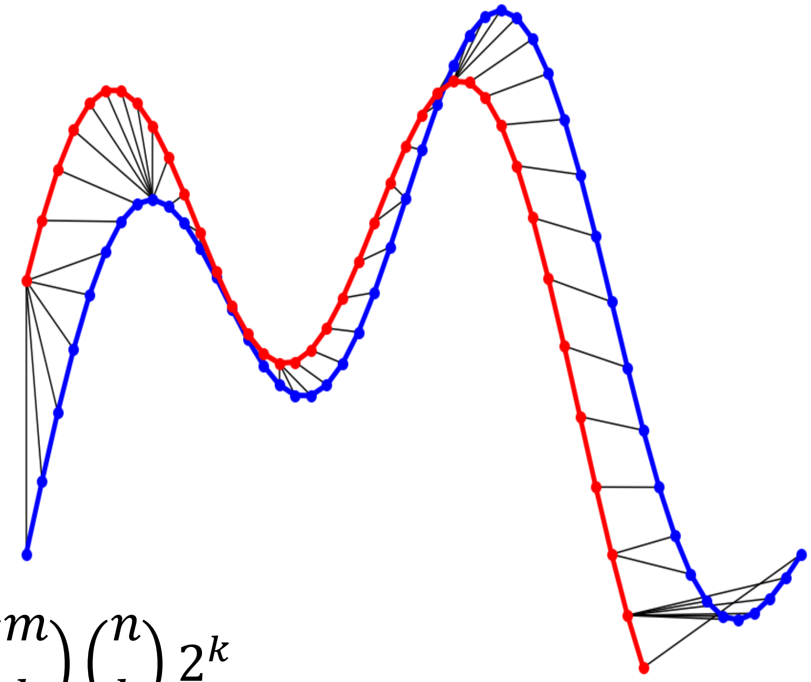
Recognition

- Classification methods compute a similarity between the input curve and models by finding a point correspondence minimizing the sum of squared distances.
- “Elastic matching” with “dynamic time warping”.
- High complexity in the number of sample points.
- Many tricks and heuristics to improve on this.

  E.g. Limit amount of dynamic time warping, pre-classify based on features, ...

Dynamic Time Warping

- Find point correspondence with minimal distance sum.



- Number of matches

$$D(m,n) = \sum_{k=0}^{\min(m,n)} \binom{m}{k} \binom{n}{k} 2^k$$

Delannoy number.

Similarity of Symbols

- Elastic matching:
  - Sequence alignment
  - Interpolation ("resampling")

- Don't we really just want the area?

Polynomial Representation

- **Main ideas**:
  - Represent strokes as parametric curves.
  - Polynomial approximation.
  - Orthogonal polynomial basis.

Polynomial Representation

- **Advantages**:
  - *Compact* – few coefficients needed
  - *Geometric*
    - – the truncation order is a property of the character set
    - – gives a natural metric on the space of characters
  - *Algebraic*
    - – properties of curves can be computed algebraically (instead of numerically using heuristic parameters)
  - *Device independent*
    - – resolution of the device is not important

Space Between Curves

$\bar{x}(t) = x(t) + \xi(t) \qquad \xi(t) \approx \sum_{i=0}^{d} \xi_i \phi_i(t), \qquad \phi_i$ orthogonal with $\langle \phi_i, \phi_j \rangle = \delta_{ij} h_j$.

$\bar{y}(t) = y(t) + \eta(t) \qquad \eta(t) \approx \sum_{i=0}^{d} \eta_i \phi_i(t)$

$$\rho^2(\bar{C}, C) \approx \langle \xi, \xi \rangle + \langle \eta, \eta \rangle$$

$$= \sum_{i=0}^{d} \sum_{j=0}^{d} \xi_i \xi_j \langle \phi_i, \phi_j \rangle + \sum_{i=0}^{d} \sum_{j=0}^{d} \eta_i \eta_j \langle \phi_i, \phi_j \rangle$$

$$= \sum_{i=0}^{d} (\xi_i^2 + \eta_i^2) \langle \phi_i, \phi_i \rangle + cross\ terms$$

$$= \sum_{i=0}^{d} (\xi_i^2 + \eta_i^2) h_i$$

Comparison of Candidate to Models

- Use Euclidean distance in the coefficient space.

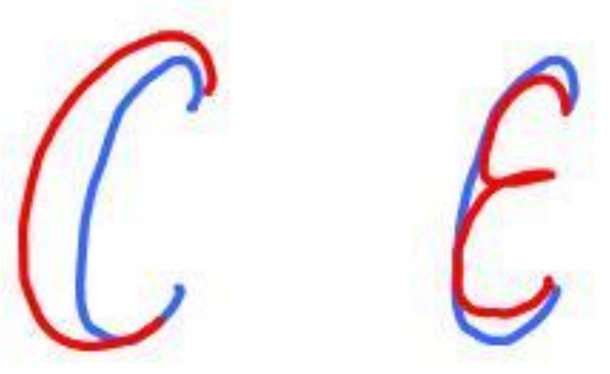
- *Just as accurate* as elastic matching.
- *Much less expensive.*
- Linear in d, the degree of the approximation.
  < 3 d machine instructions (30ns) *vs* several thousand!
- Can trace through SVM-induced cells incrementally.

- Normed space for characters gives other advantages.

Problems

- Want fast response –
  how to work while trace is being captured.

- Low separation area does not mean similar shape.



Problem 1. On-Line Ink

- A problem:

  In handwriting recognition, the human and the computer take turns thinking and sitting idle.

- We ask:

  Can we do useful work while the user is writing and thereby get the answer faster after the user stops writing?

- *The answer is "Yes"!*

On-Line Series Coefficients

- Use Legendre polynomials P_i as basis on the interval $[-1,1]$, with weight function 1.
- Collect numerical values for $f(\lambda)$ on $[0, L]$.
  $\lambda \quad =$ arc length.
  L is not known until the pen is lifted.
- *As the sample points are collected*, numerically integrate the moments $\int \lambda^i f(\lambda) d\lambda$.
- After last point, compute series coefficients for f
  with domain and range scaled to $[-1,1]$.
  This uses a simple linear transformation of the moments.

On-Line Series Coefficients

- Transform moments $\mu_i(f, L)$ of $f(\lambda)$ on $[0, L]$ to coefficients of $\hat{f}(\lambda) = \sum_k \hat{\alpha}_k P_k(\lambda)$ on $[-1, 1]$:

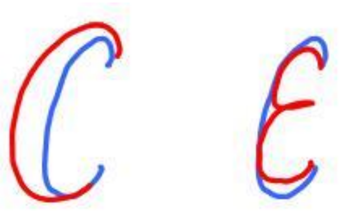
$$\hat{\alpha}_k = (-1)^k \frac{2k+1}{L} \sum_{i=0}^{k} \left(\frac{-1}{L}\right)^i \binom{k}{i} \binom{k+i}{i} \mu_i(f, L)$$

- Normalize range of f:

$$\hat{\alpha}_k \frac{b-a}{f_M - f_m} + \delta_{i0} \frac{a\, f_M - b\, f_m}{f_M - f_m}$$

- Constant time at pen up.

Problem 2. Shape vs Variation



- The corners are not in the right places.
- Work in a jet space to force coords & derivatives close.
- Use a Sobolev inner product

$$\langle f, g\rangle = \int_a^b f(t)g(t)w_0(t)dt$$

$$+\mu_1 \int_a^b f'(t)g'(t)w_1(t)dt + \mu_2 \int_a^b f''(t)g''(t)w_2(t)dt + \cdots$$

- 1^{st} jet space ⇒ set $\mu_i = 0$ for $i > 1$.
  - Choose μ_1 experimentally to maximize reco rate.
  - Can be also done on-line.

    [Golubitsky + SMW 2008, 2009]

Linear Separability

- Can separate N classes with $N(N-1)$ SVM planes.
- Each class is then (mostly) within its own convex polyhedral cell.
- Can classify either by
  - SVM majority voting + run-off elections (96%)
  - Distance to convex hull of *k* nearest neighbours (97.5%).
  - On-line computation.

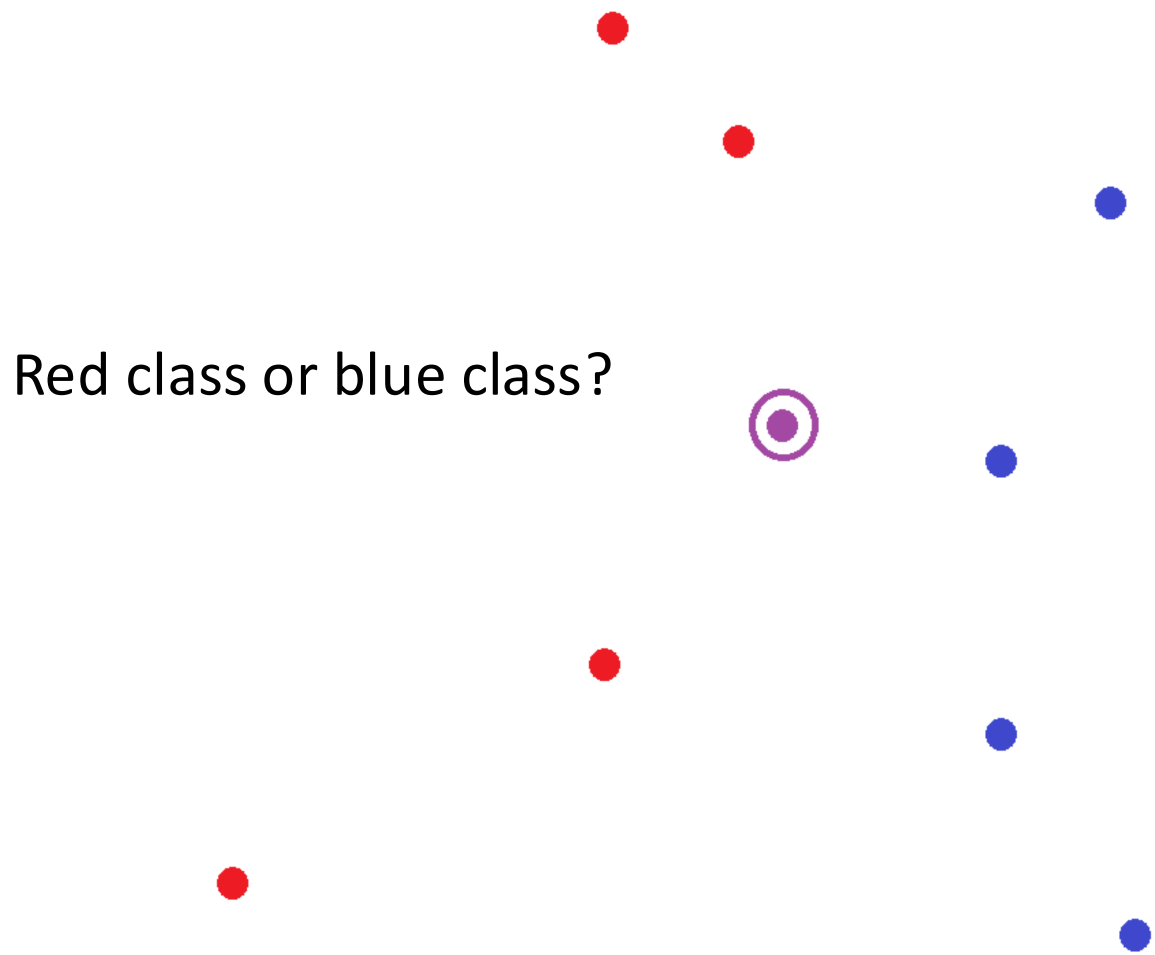
The Joy of Convexity

- Convexity ⇒ Linear homotopies stay within a class

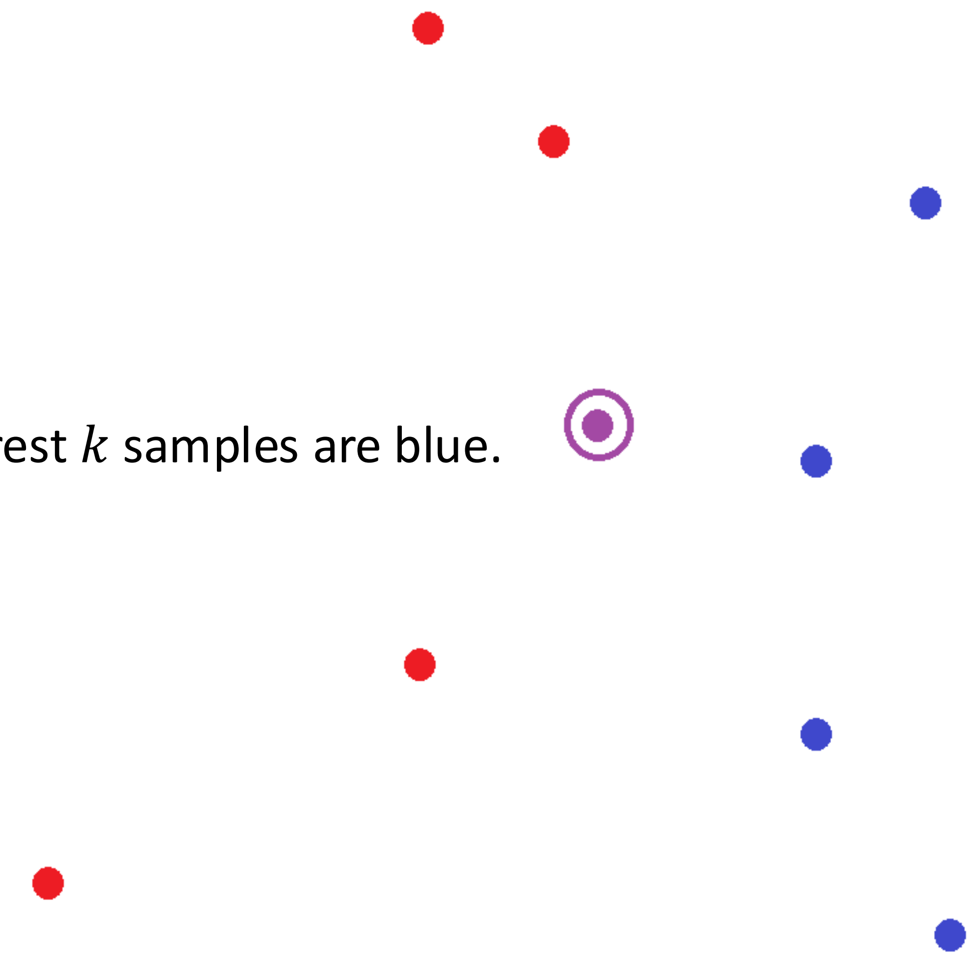
$$C = (1- t)\, A + t\, B$$

- Can compute distance of a sample to this line
- Distance to convex hull of nearest neighbors in class gives best recognition [Golubitsky+SMW 2009,2010]

Choosing Among Alternatives



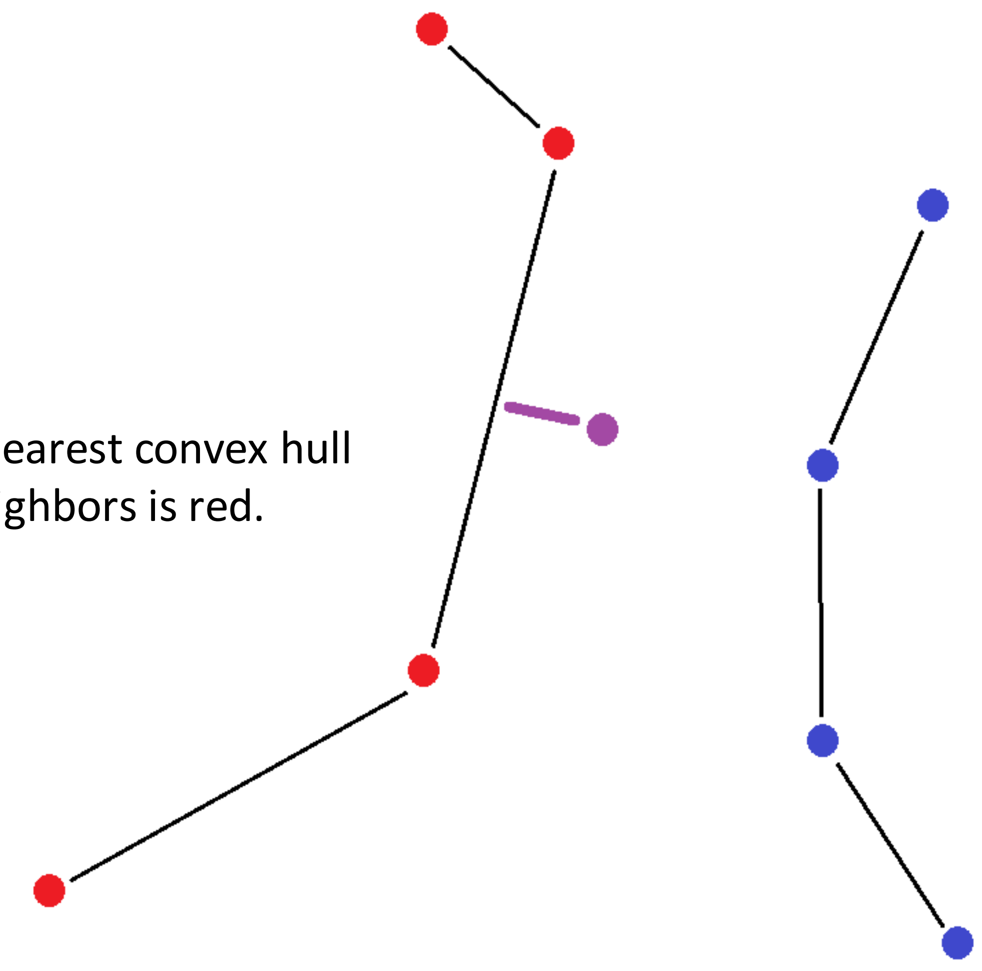
Choosing between Alternatives



The nearest k samples are blue.

Choosing between Alternatives

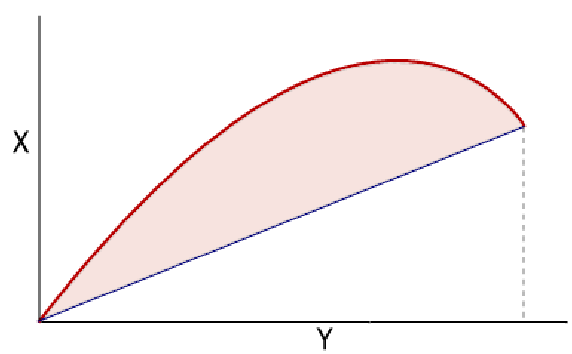
The nearest convex hull of neighbors is red.



Orientation and Shear

- Reco when writing at an angle, or with slanted chars.

- Instead of taking ortho series of coord fns $x(\lambda)$ and $y(\lambda)$, use ortho series of integral invariants of these.
  [Golubitsky, Mazalov, SMW 2009 rotn, 2010 shear]



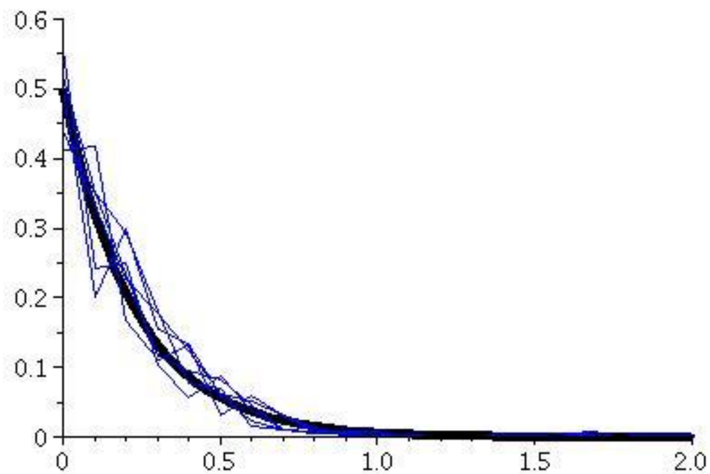
$I_0(\lambda) = radius$ $\qquad$ $I_1(\lambda) = area$

$I_{k>1}(\lambda) = more\ complicated\ integrals$

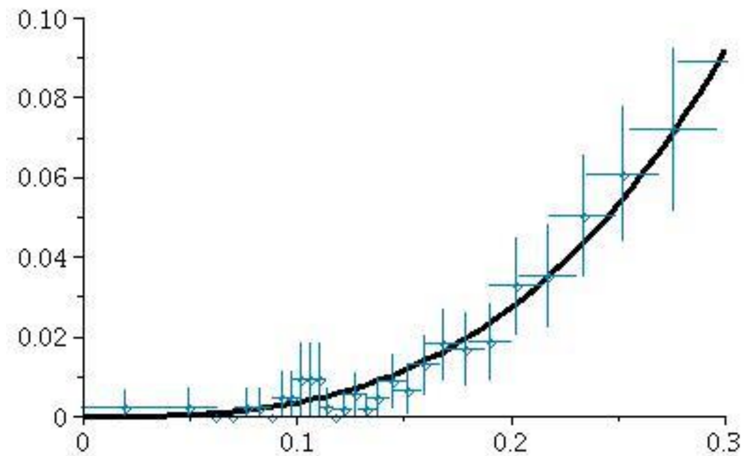
Training

- Using CHKNN allows training with relatively few samples. (Dozens vs Thousands per class)

Error Rates as Fn of Distance



SVM



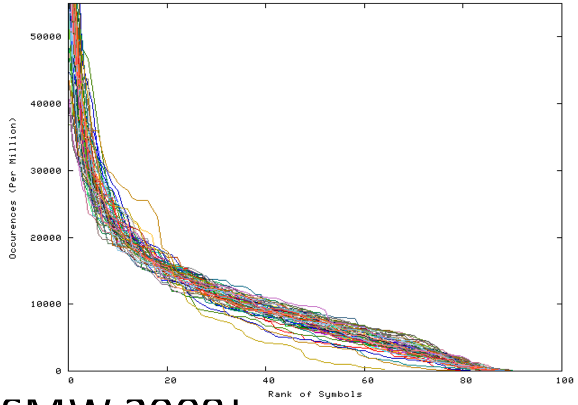
Convex Hull

- Error rate as fn of distance gives confidence measure for classifiers [MKM – Golubitsky + SMW 2009]

Combining with Statistical Info

- Empirical confidence on classifiers allows geometric recognition of isolated symbols
  to be combined with statistical methods.

- Domain-specific *n*-gram information:
  - Research mathematics –
    20,000 articles from arXiv
    [MKM -- So+SMW 2005]
  - 2nd year engineering math –
    most popular textbooks
    [DAS -- SMW 2008]
  - Inverse problem –
    identifying area via *n*-gram freq! [DML -- SMW 2008]



Deciding with Confidence Measure

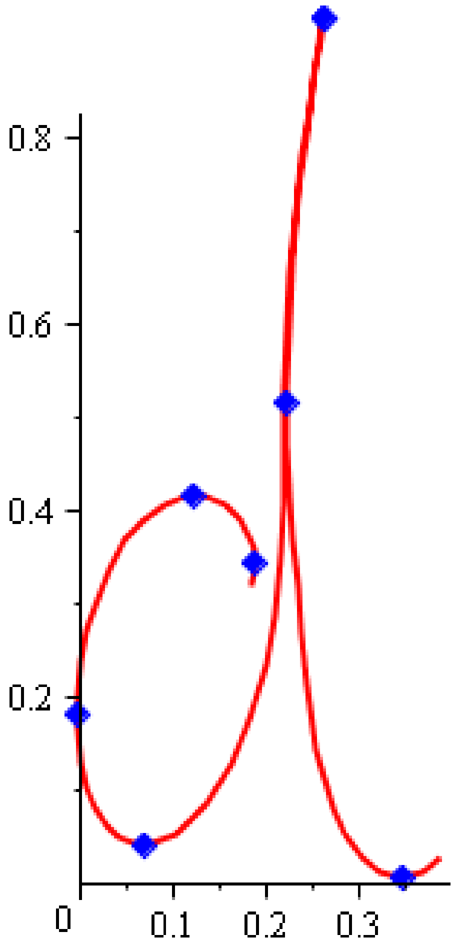
Symbol X in an Expression E

Symbol Recognizer:
$X \in \text{Class}_1$ **with Conf** ξ_1

Context-Based Predictor:
$X \in \text{Class}_2$ **with Conf** ξ_2

$X \in \text{Class}_i \Leftrightarrow \xi_i = \max(\xi_1, \xi_2)$

Sensible Critical Points



- Functional approx uses non-local information.
- Puts critical points where they should be.
- Polynomial common root finding.

$$\frac{dx}{d\lambda} = 0 \quad \frac{dy}{d\lambda} = 0$$

SNC Problems

- Symbolic Numeric Algorithms for Polynomials
- Step away from fuzzy equality in algos for exact computation

- "The Singular Value Decomposition for Polynomial Systems"
  Corless, Gianni, Trager, Watt 1995

  Introduced backward error analysis to symbolic computation:

  E.g. Given two polynomials $p, q \in Z[x]$ of degree $\leq d_{pq}$,
  do there exist $\Delta p, \Delta q$ of degree $\leq d$ and norm $\leq \varepsilon$ such that
  $\gcd(p + \Delta p, q + \Delta q)$ has degree $\geq d_g$ and, if so, find them.

SNC Problems for HWR

- Transformation in and out of monomial basis is ill-conditioned.
- Want to compute resultants, *etc*, without transforming to monomial basis
  - work with Alvandi [Legendre Sobolev] and
  - with Singh Kalhan [Chebyshev-Sobolev]

Real Time Capture

Proposition 1. *Let* $m_k(f,L) = \int_0^L \lambda^k f(\lambda) \mathrm{d}\lambda$ *and* $\hat{f}(\lambda) = \sum_{k=0}^{d} \alpha_k S_k^{\mu}(\lambda)$ *be the scaled function on* $[-1,1]$*. Then* α_k *may be computed as*

$$\alpha_k = \sum_{j=0}^{d} \frac{m_j(f,L)}{L^{j+1}} \; C_{j+1 \;\; k+1}$$

$$C_{ij} = \begin{cases} (-1)^{i+j} c_{ij}(\mu) & \textit{for } i > j \\ (-1)^{i+j} c_{ij}(\mu) + (-1)^{i+j}(2j-1)\binom{j-1}{i-1}\binom{i+j-2}{j-1}\frac{1}{a_{j-1}(\mu)} & \textit{for } i \leq j \end{cases}$$

$$c_{ij}(\mu) = \left(\frac{1}{a_{j-1}(\mu)} - \frac{1}{a_{j+1}(\mu)}\right) \sum_{k=\max(1,\lceil \frac{i-j}{2} \rceil)}^{\lfloor \frac{d+1-j}{2} \rfloor} (2j+4k-1)\binom{j+2k-1}{i-1}\binom{i+j+2k-2}{j+2k-1}$$

$$a_h(\mu) = \sum_{k=0}^{\max(0,\lfloor \frac{h-1}{2} \rfloor)} \left(\frac{\mu}{4}\right)^k \frac{(h+2k-1)!}{(2k)!(h-2k-1)!}.$$

Note that the entries of matrix C are independent of the problem and may be computed in advance.

with Parisa Alvandi

Derivatives

Theorem 2 *We have*

$$\left[\, S_1'(\lambda) \cdots S_n'(\lambda) \,\right]^{\boldsymbol{T}} = D \left[\, S_0(\lambda) \cdots S_{n-1}(\lambda) \,\right]^{\boldsymbol{T}},$$

where $D = H \times N^{-1}$, *and for* $i, j = 1, \ldots, n$, *we have*

$$[D]_{i,j} := \begin{cases} \frac{(2\,i-1)a_i(\mu)}{a_{i-1}(\mu)} & \text{if } i = j \\ \frac{(2\,j-1)a_j(\mu)}{a_{j-1}(\mu)} + (\frac{1}{a_{j-1}(\mu)} - \frac{1}{a_{j+1}(\mu)})\times & \text{if } j = i - 2\,\ell, \\ \sum_{k=0}^{\lfloor \frac{i-j}{2} \rfloor - 1} (2\,i - 4\,k - 1)a_{i-2k}(\mu) & \text{for } \ell = 1, \ldots, \lfloor \frac{i-1}{2} \rfloor \\ 0 & \text{otherwise.} \end{cases}$$

GCD

Theorem 7 *Let* $f(\lambda) = \sum_{i=0}^{n} \alpha_i S_i^{\mu}(\lambda)$ *and* $h(\lambda) = \sum_{i=0}^{m} \beta_i S_i^{\mu}(\lambda)$, *where* $n > m$, *and matrix* B_n *is the matrix defined in Theorem* 6 *for polynomial* $f(\lambda)$. *Let also* $g(\lambda)$ *be the monic gcd of polynomials* $f(\lambda)$ *and* $h(\lambda)$ *and* $g(\lambda) := \lambda^k + \sum_{i=0}^{k-1} \hat{\zeta}_i \lambda^i = \sum_{i=0}^{k} \zeta_i S_i^{\mu}(\lambda)$. *Define* $h(B_n) = \beta_0 I_n + \beta_1 S_1^{\mu}(B_n) + \cdots + \beta_m S_m^{\mu}(B_n)$, *then* $k = n - rank(h(B_n))$. *Furthermore, let* c_i *be the* i*th column of matrix* $h(B_n)$, *for* $i = 1, \dots, n$. *Then the columns* $c_{k+1}, \dots, c_n$ *are linearly independent and if the numbers* $x_{i,j}$ *are defined by*

$$c_i = x_{i,k+1} c_{k+1} + \sum_{j=k+2}^{n} x_{i,j} c_j, \quad \text{for } i = 1, \dots, k, \quad \text{and some } x_{i,j} \in \mathbb{R} \tag{15}$$

then $\zeta_{i-1} = \zeta_k x_{i,k+1}$, *for* $i = 1, \dots, k$, *where* $\zeta_k := (\prod_{\ell=1}^{k} [B_n]_{\ell,\ell+1})^{-1}$.

Can be computed with SVD method [Corless, Gianni, Trager + W 1996]

Bounding Sobolev Norms

Theorem

Let f and g be two polynomials of degree $\leq d$ with coefficient vectors $\mathbf{f}$ and $\mathbf{g}$ in the polynomial basis $\{B_i\}_{i=0}^{d}$ orthogonal wrt the inner product $\langle p, q \rangle_S = \int_0^1 p(t)q(t)w_0(t)dt + \int_0^1 p'(t)q'(t)w_1(t)dt$, and let $\mathbf{D}$ be the differentiation matrix for the basis.

Then the Sobolev norm of the difference between f and g satisfies

$$\|f - g\|_S \leq \sqrt{d+1}\, \|\mathbf{f} - \mathbf{g}\|_\infty (1 + \mu \|\mathbf{D}\|).$$

with Deepak Singh Kalhan

Differentiation Matrices

- Polynomial differentiation is a linear operation so may be represented matrix multiplication on coefficient vectors.
- For basis $\{B_i\}_{i=0}^{d}$, define $\mathbf{D}$ by $B_i'(x) = \sum_{j=0}^{d} D_{ji} B_j(x)$.
- Then $\mathbf{f}' = \mathbf{D}\,\mathbf{f}$, for $f(x) = \sum_{i=0}^{d} f_i B_i(x)$, $f'(x) = \sum_{i=0}^{d} f_i' B_i(x)$.
- Degree 5 examples:

$$\mathbf{D}_{x^i} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 2 & 0 & 0 & 0 \\ & & 0 & 3 & 0 & 0 \\ & & & 0 & 4 & 0 \\ & & & & 0 & 5 \\ & & & & & 0 \end{bmatrix} \qquad \mathbf{D}_{T_i} = \begin{bmatrix} 0 & 1 & 0 & 3 & 0 & 5 \\ & 0 & 4 & 0 & 8 & 0 \\ & & 0 & 6 & 0 & 10 \\ & & & 0 & 8 & 0 \\ & & & & 0 & 10 \\ & & & & & 0 \end{bmatrix}$$

[Amiraslani,C,Gunasingham 2000]

Baseline Estimation

- Figure out baseline from the characters,
rather than the other way around, which is more ususal.

- We can locate some important features by identifying special points.

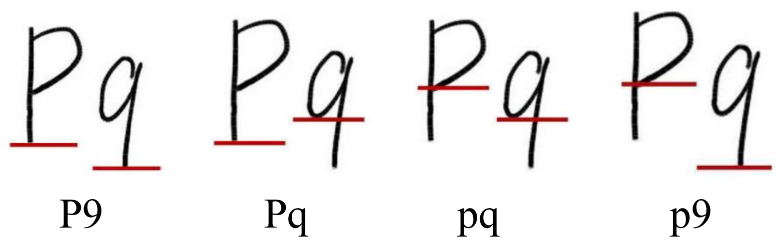
We refer to a point such as this, that determines the height of a metric line, as a *determining point*.

Determining Points

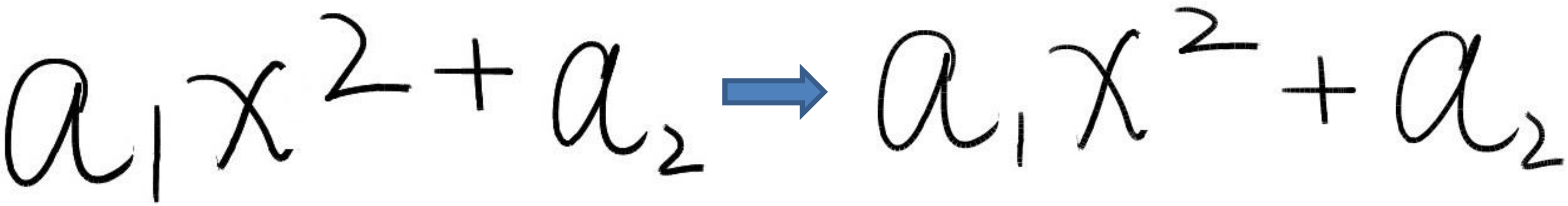
- We focus on European alphabets.
- We consider 6 types of determining points.

Baseline Estimation

- Juxtaposition ambiguity



- Handwriting neatening

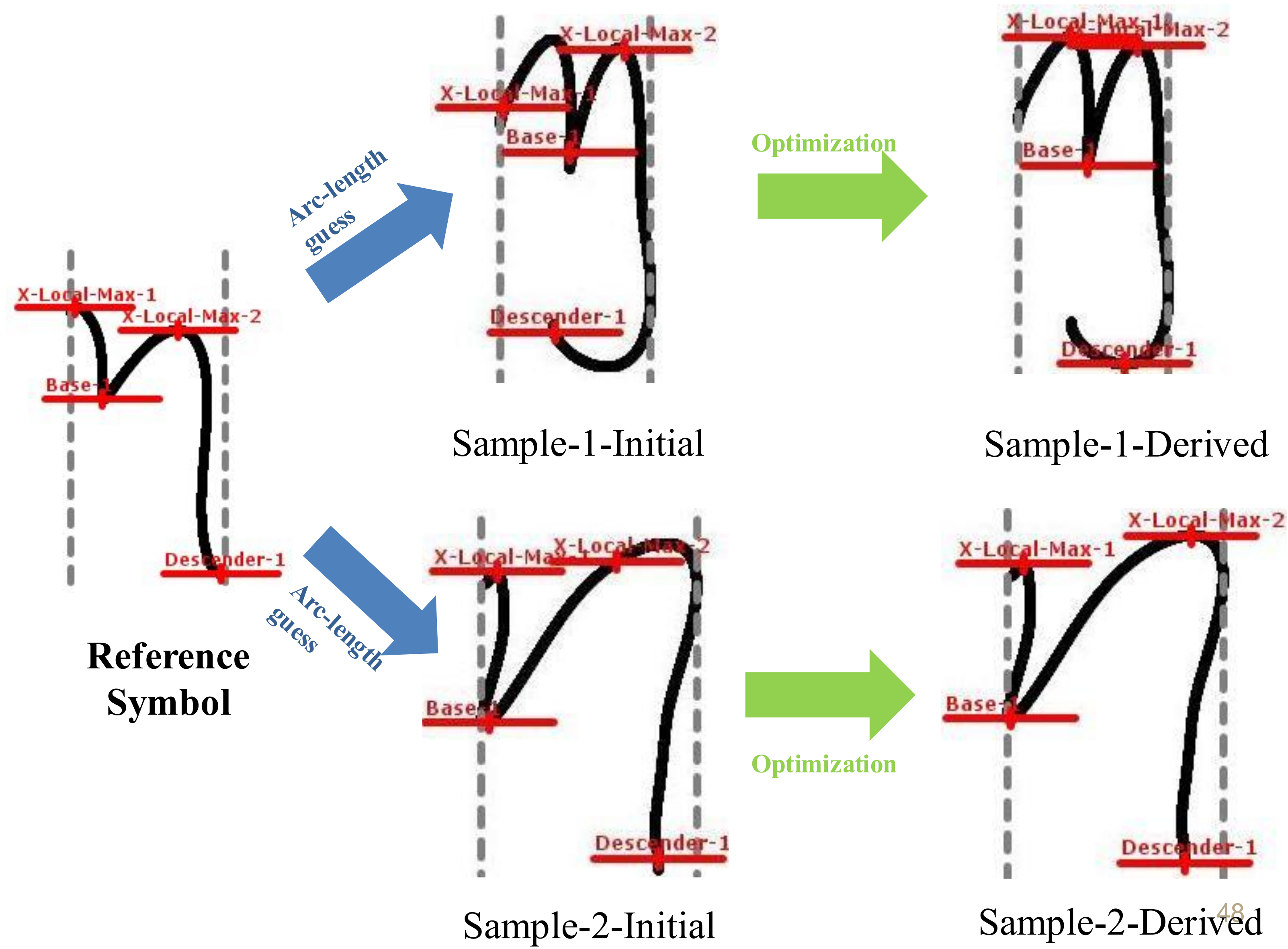


Average Symbol

- The average symbol of a set of known samples for a class can be computed as the average point in the functional space,

$$\bar{C} = \sum_{i=1}^{n} C_i / n$$

Deriving from a Reference Symbol

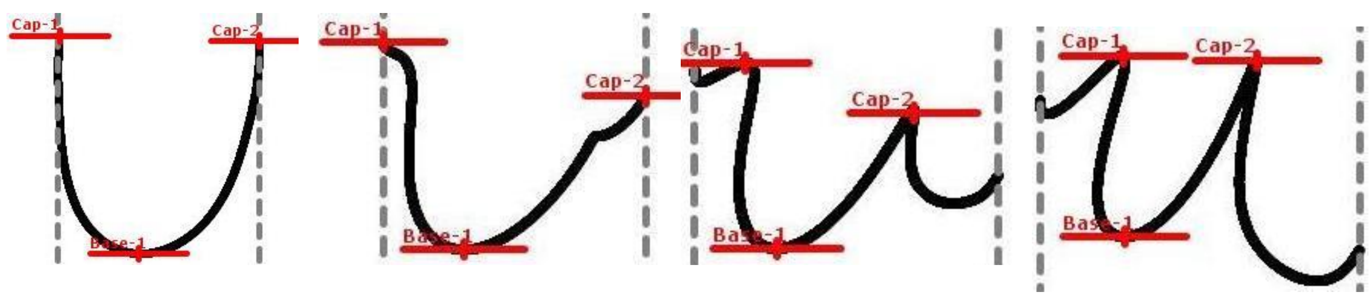


A Homotopy Method

- Some samples are far away from the reference symbol.

Average **Target**

- We use a homotopy between the reference symbol and the target sample in a multi-step method.

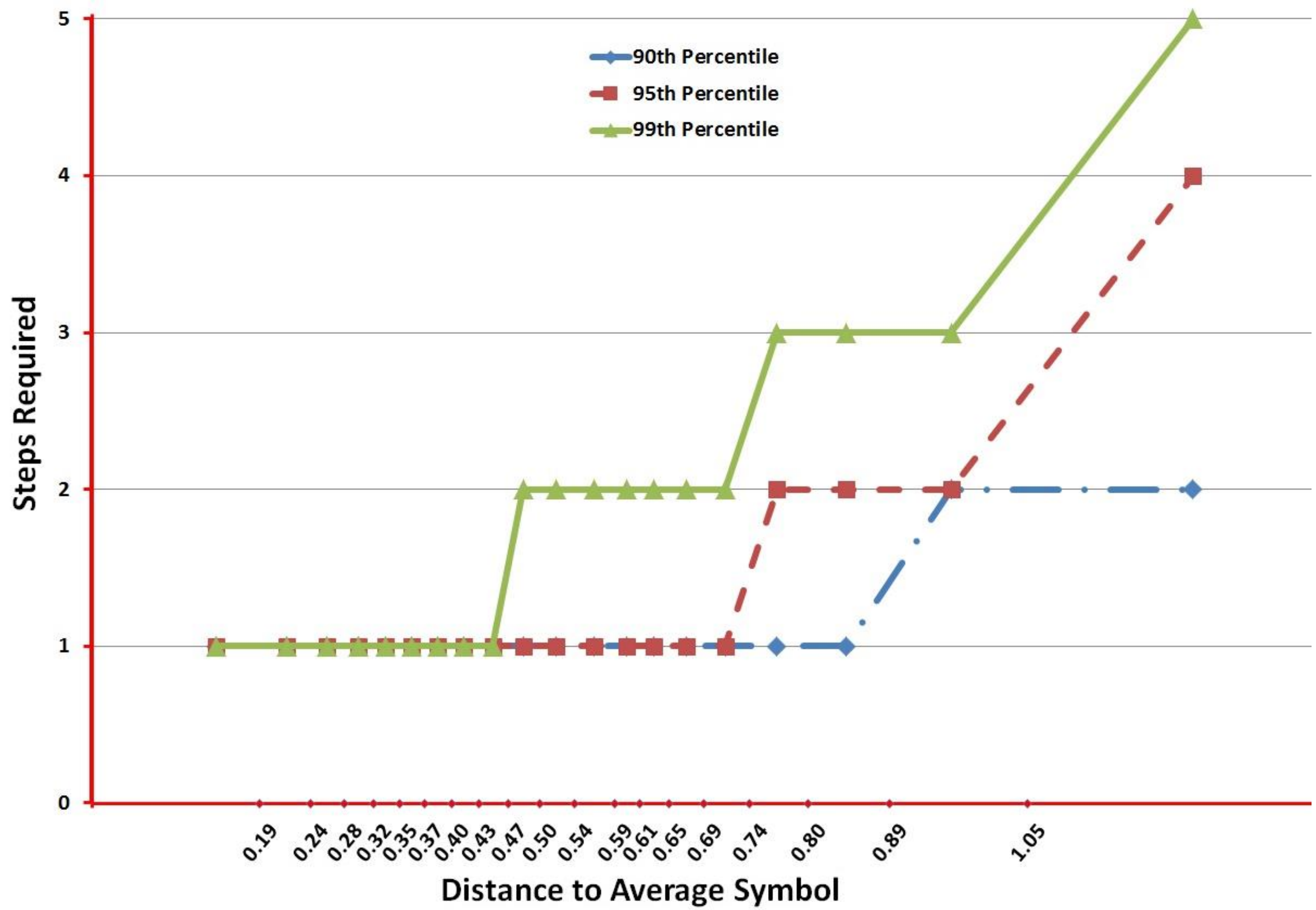


Average **Step-1** **Step-2** **Target**

Measuring Number of Steps Needed

- We went back and tested the data by forgetting the marked points and seeing how many steps it took to recover them.
- For each sample, we tried 1 step, 2 steps, etc up to 30, until a correct result was achieved.
- Samples that did not converge with a 30 step homotopy were considered to fail.
- We examined the ***relationship*** between the ***distance*** to the reference symbols and the ***number of steps*** the homotopy required.

Number of Steps as Fn of Distance to Average Symbol



The overall success rate is 99.63%. Each distance interval in the dense area (x≤0.59) contains 2800 samples while each interval in the sparse area (x>0.59) contains 1500 samples except the last one contains 1370 samples.

Recognition Summary

- Database of samples ⇒ set of LS or ChS points
- Character to recognize ⇒
  - reparameterize by arc-length
  - numerically integrate to get coef. vector
- Classify by distance to convex hull of k-NN.
- Very fast
- Requires very few training points

Conclusions

- Ask what are we really trying to do.
- Work with ink traces as ***curves***,
  rather than as collections of sample points.

- Admits powerful analytic tools.
- Have useful geometry on space of curves.

- Gives device/resolution independence.
- Gives useful insights and faster algorithms.

- Needs only a few examples of each class.
- Compact representation of reference samples.

Conclusions

- The handwriting application is one instance of

  curve classification can improve when we use its geometry rather than converting it too early into pixels, point clouds, or ad hoc features.