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# Preconditioning and Linearly Implicit Time Integration for the Serre-Green-Naghdi Equations

David Shirokoff

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In Collaboration with: Wooyoung Choi (NJIT)  
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DMS-1719693

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# Outline

1. Introduction to Serre-Green-Naghdi (SGN) Equations.
2. Preconditioner for SGN Constraint Equation.
3. Semi-Implicit Time Integration for SGN.

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# Part I

## Introduction to Serre-Green-Naghdi (SGN) Equations.

# Fluids with a Free Surface



Hurricane Sandy licensed under [CC BY-NC-ND](https://creativecommons.org/licenses/by-nc-nd/4.0/)

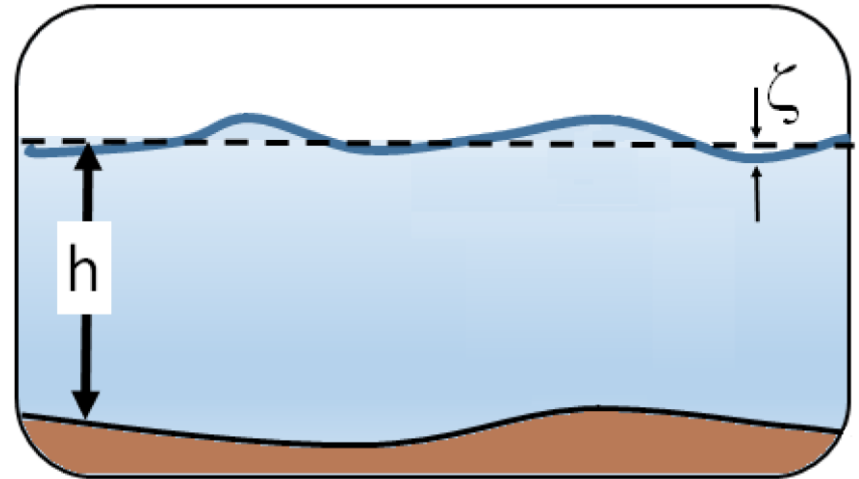


# Incompressible Euler Equations

Domain 3d:  $(x, z) \in \mathbb{R}^{2+1}$

Fluid variables:  $\mathbf{w} = (\mathbf{u}, w)$

Pressure:  $p$



Fluid model (no dissipation, incompressible):

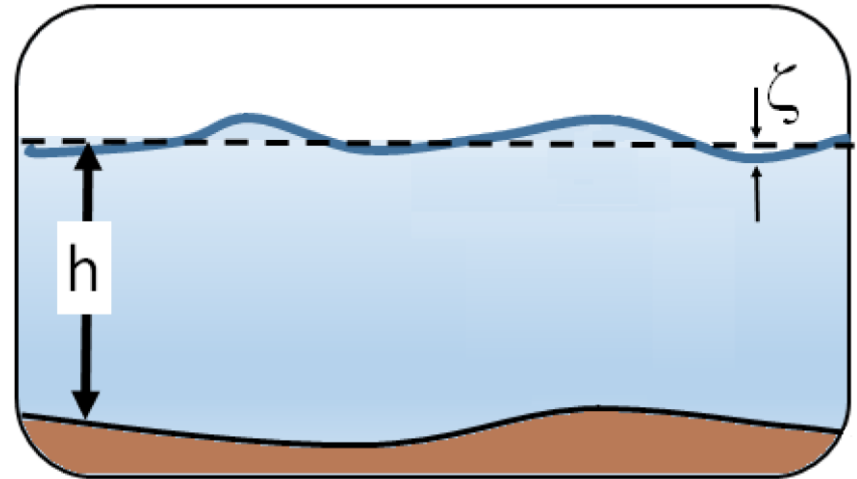
$$\mathbf{w}_t + \mathbf{w} \cdot \nabla \mathbf{w} = -\frac{1}{\rho} \nabla p - \mathbf{g}, \quad \text{in } -h(x, y) < z < \zeta(x, y, t)$$
$$\nabla \cdot \mathbf{w} = 0.$$

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Boundary conditions (no flux, no surface tension):

$$w = 0 \quad \text{on } z = -h(x, y)$$

$$\begin{aligned} \zeta_t + \mathbf{u} \cdot \nabla \zeta &= w \\ p &= 0 \end{aligned} \quad \text{on } z = \zeta(x, y, t)$$

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**UNKNOWN(S) (TO BE SOLVED):  $w$ ,  $p$  AND  $\zeta$ !**

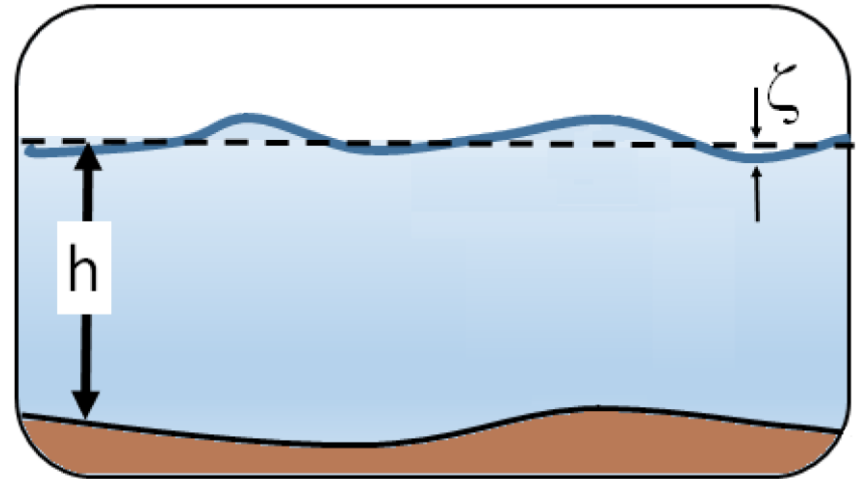
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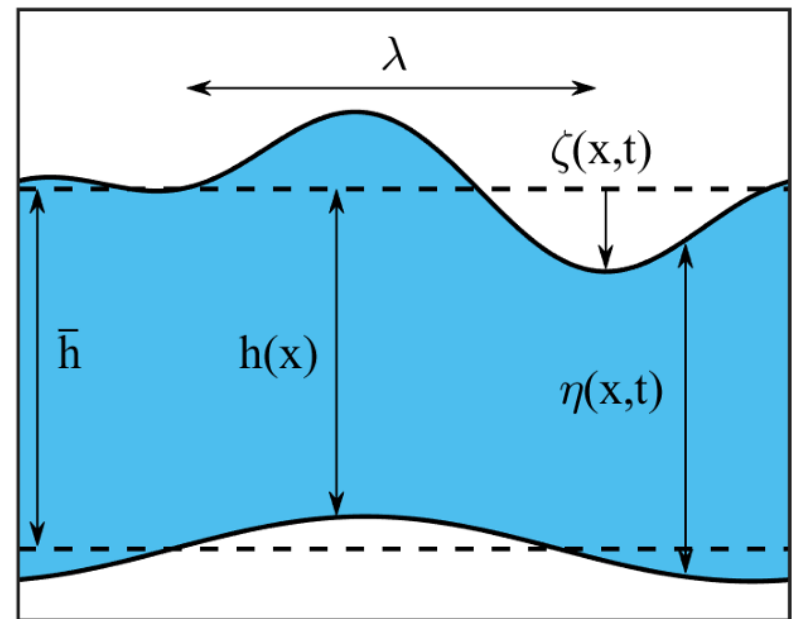
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# Quest for Simpler Models

- Incompressible Euler is a PDE (in 3d!) on a time dependent domain.
- Solving for the domain (free surface or interface) is part of the problem.
- “Simpler” models represent free surface as a graph over a 2d fixed domain.



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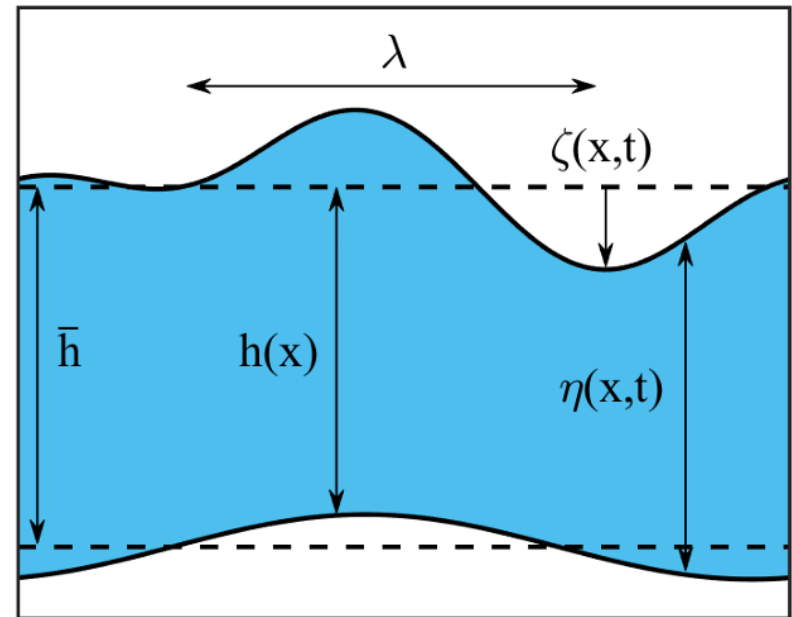
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Length scales for typical problem:

$$\beta = \bar{h}/\lambda \quad \text{-water depth v. wavelength}$$

$$\alpha = a/\bar{h} \quad \text{-amplitude v. water depth}$$

$$a = O(\zeta)$$



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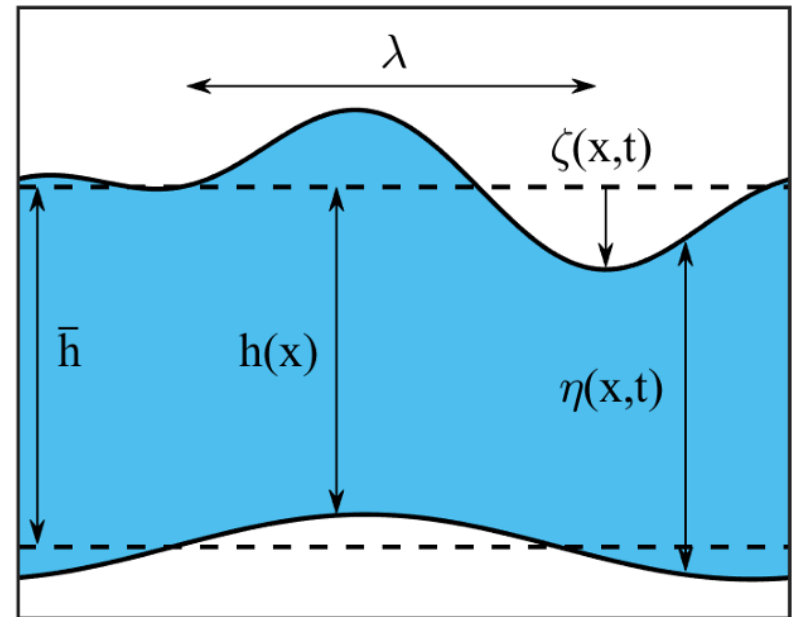
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Shallow Water Assumption:

$$\beta \ll 1$$



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# The Shallow Water Equations

Under the Shallow Water Assumption:  $\beta := \bar{h}/\lambda \ll 1$

Introduce depth averaged velocity:

$$\bar{\mathbf{u}}(\mathbf{x}, t) = \frac{1}{\zeta + h} \int_{-h}^{\zeta} \mathbf{u}(\mathbf{x}, z, t) \, dz$$

Where the velocity has components in 2d:

$$\bar{\mathbf{u}}(\mathbf{x}, t) = (\bar{u}(x, y, t), \bar{v}(x, y, t)) \quad \text{and} \quad \mathbf{x} := (x, y) \in \Omega \subseteq \mathbb{R}^2$$

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Now:

- Depth average the Euler equations, and
- Perform (formal) asymptotic expansion in  $\beta$ .

At leading order, *Shallow Water Equations*:

$$\begin{aligned} \eta_t + \nabla \cdot (\eta \bar{\mathbf{u}}) &= 0 \\ \bar{\mathbf{u}}_t + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} + g \nabla \zeta &= O(\beta^2) \end{aligned} \quad \text{for } (x, y) \in \Omega$$

Where, the water depth is:

$$\eta(\mathbf{x}, t) = h(\mathbf{x}) + \zeta(\mathbf{x}, t)$$

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Computational advantages:

- PDE is now only 2d instead of 3d; alleviates the curse of dimensionality;
- Free surface appears as dependent PDE variable;
- Domain is fixed in time.
- Large scale solutions since the 1980's.

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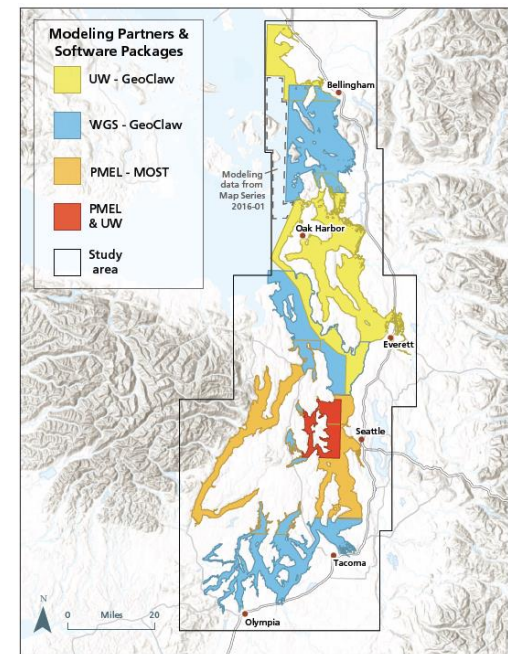
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Remarks:

- Used to model "large" domains, e.g., oceans and coastal regions.
- No small amplitude assumption required,  $\alpha = a/\bar{h} = O(1)$

Right: GeoClaw (Clawpack)

Image from: "Tsunami Hazard Maps of the Puget Sound and Adjacent Waters" Washington Geological Survey, April 2021.



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# Incorporation of Dispersion

More Remarks:

- Shallow Water Equations are hyperbolic.
- Solutions may contain shock waves.
- Linearized SWE (e.g., short time, small amplitude solutions) constant bathymetry have a linear dispersion relation:

$$\eta \approx h + \zeta_k e^{i(\omega t + \mathbf{k} \cdot \mathbf{x})} \quad \mathbf{u} \approx \mathbf{U}_k e^{i(\omega t + \mathbf{k} \cdot \mathbf{x})} \quad \text{where} \quad \omega^2 = gh|\mathbf{k}|^2$$

SWE (dropping averages):

$$\begin{aligned} \eta_t + \nabla \cdot (\eta \mathbf{u}) &= 0 \\ \underbrace{\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \zeta}_{\text{SWE terms}} &= 0 \end{aligned}$$

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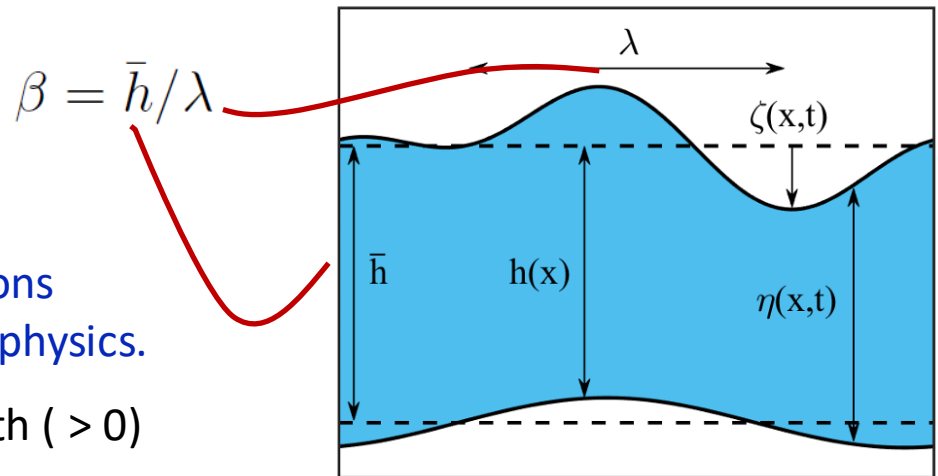
SWE (dropping averages):

$$\underbrace{u_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \zeta}_{\text{SWE terms}} = \boxed{O(\beta^2)} \quad \text{Higher corrections} \\ \text{= more physics.}$$

$$\eta(\mathbf{x}, t) = h(\mathbf{x}) + \zeta(\mathbf{x}, t) \quad \text{– water depth ( > 0)}$$

$$h(\mathbf{x}) \quad \text{– bathymetry}$$

$$\zeta(\mathbf{x}, t) \quad \text{– free surface height}$$



$\mathbf{u}(\mathbf{x}, t)$  – depth avg. velocity (2d)

# Serre-Green-Naghdi Equations (SGN)

See: [Rayleigh 1876]; [Serre 1953];  
[Su, Gardner 1969]; [Green, Naghdi 1976]

Shallow water equation  
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$O(\beta^2)$  (truncated to order 4)

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- Shallow water equations (SWE) hyperbolic – contain shocks.
- SGN – add dispersion effects to SWE (which can regularize shock) can create additional oscillations, e.g., more waves; valid for O(1) nonlinearities (not just weakly nonlinear).

## Linearization (about constant state):

$$\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{u}} e^{i\omega t - i\mathbf{k} \cdot \mathbf{x}}$$

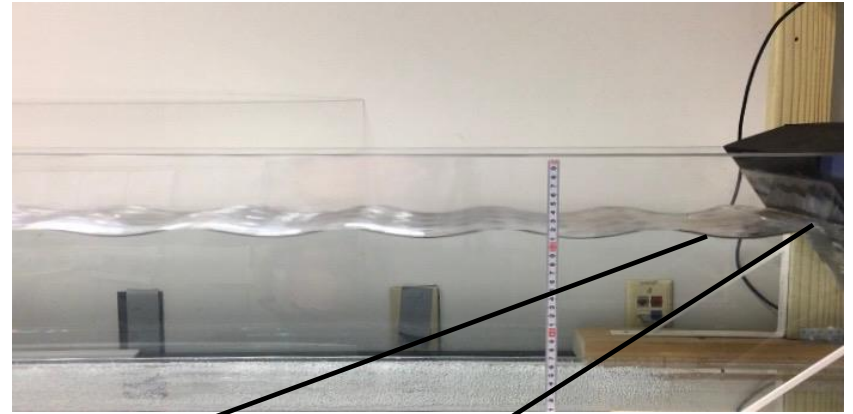
$$\eta(\mathbf{x}, t) = h_0 + \hat{\zeta} e^{i\omega t - i\mathbf{k} \cdot \mathbf{x}}$$

Dispersion  
relation  
→

$$\omega = \frac{|\mathbf{k}|}{\sqrt{1 + |\mathbf{k}|^2}}$$

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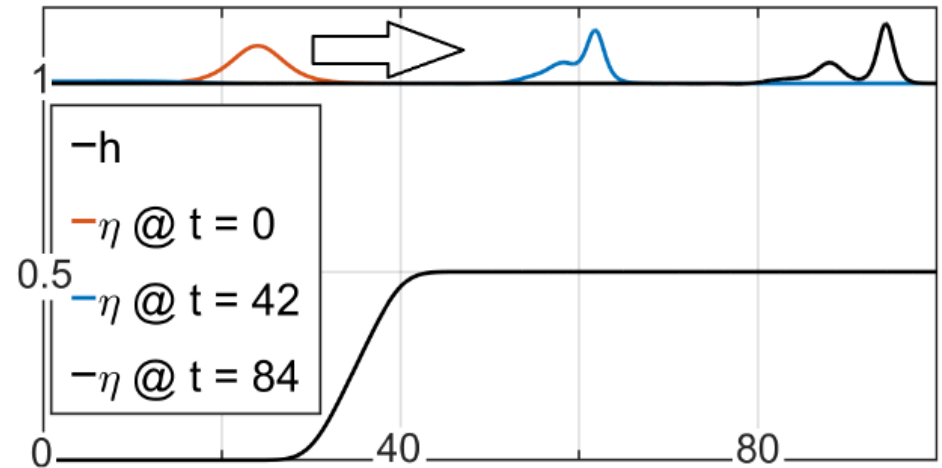
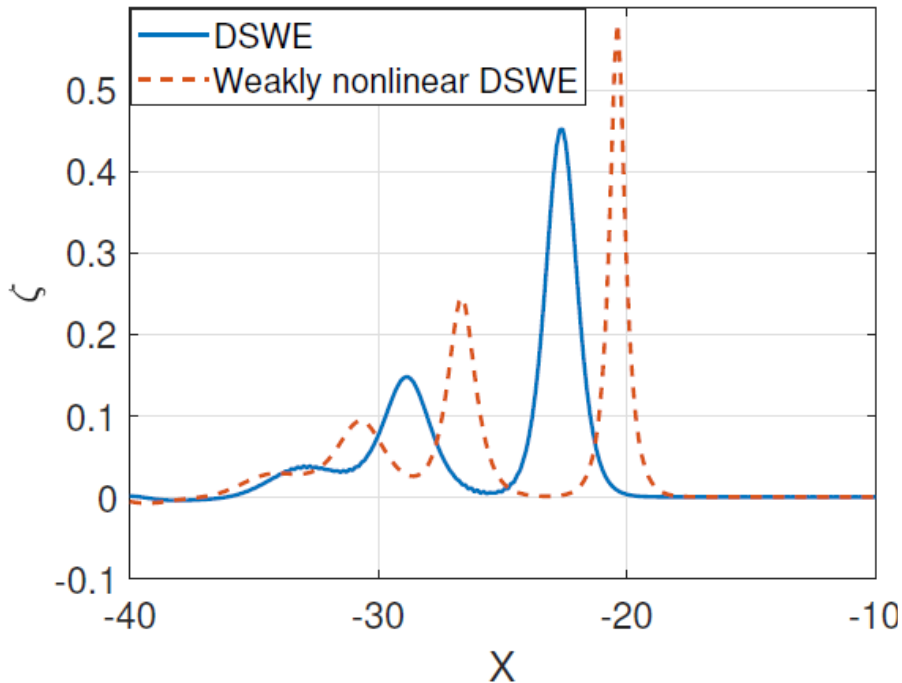
# Wave Tank Experiments



Courtesy of Wooyoung Choi's Lab – NJIT,  
Department of mathematical science

# Classic Example: Role of dispersion

Dispersion can act to regularize a shock – creating oscillations and a wave train of solitons.



Numerical solutions via schemes to come.

For flat bottom, exact (family) soliton solutions:

$$\eta = h + a \operatorname{sech}^2(\gamma\xi) \quad \mathbf{u} = (u, v) = w(\cos \theta, \sin \theta) \quad (\gamma h)^2 = \frac{a}{h+a} \quad w = c \left(1 - \frac{h}{\eta}\right) \quad \frac{c^2}{gh} = 1 + \frac{a}{h}$$

$$\xi = x \cos(\theta) + y \sin(\theta) - ct$$

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# What are the numerical challenges?

# Serre-Green-Naghdi Equations (SGN)

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## Numerics

Difficulty 1:

**Nonlinear**, and “Mixed” space-time derivative terms, e.g.,  $\sim \eta^{-1} \nabla (\eta^3 \nabla \cdot \mathbf{u}_t)$

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**Nonlinear**, and “**Mixed**” space-time derivative terms, e.g.,  $\sim \eta^{-1} \nabla (\eta^3 \nabla \cdot \mathbf{u}_t)$

Difficulty 2?

**RHS has term**  $\eta^{-1} \nabla (\mathbf{u} \cdot \nabla (\nabla \cdot \mathbf{u}))$ , explicit treatment could be very stiff?

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# SGN Constraint

$$\eta_t + \nabla \cdot (\eta \mathbf{u}) = 0,$$

$$\mathcal{G}u_t + \eta \mathbf{u} \cdot \nabla \mathbf{u} + \eta \nabla \zeta = \nabla \left\{ \frac{1}{3} \eta^3 \left( \mathbf{u} \cdot \nabla (\nabla \cdot \mathbf{u}) - (\nabla \cdot \mathbf{u})^2 \right) + \frac{1}{2} \eta^2 (\mathbf{u} \cdot \nabla)^2 h \right\} \\ - \eta \left\{ \frac{1}{2} \eta \left( \mathbf{u} \cdot \nabla (\nabla \cdot \mathbf{u}) - (\nabla \cdot \mathbf{u})^2 \right) + (\mathbf{u} \cdot \nabla)^2 h \right\} \nabla h,$$

Bring all time derivatives to left hand side, where

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Bring all time derivatives to left hand side, where

$$\mathcal{G} \equiv \mathcal{N}_\eta + \mathcal{K}_{\eta,h}: \begin{cases} \mathcal{N}_\eta \equiv \eta \mathbf{I} - \nabla \frac{1}{3} \eta^3 \nabla \cdot, & \text{(dispersion due to the surface)} \\ \mathcal{K}_{\eta,h} \equiv -\nabla \left( \frac{1}{2} \eta^2 \nabla h^T \cdot \right) + \frac{1}{2} \eta^2 \nabla h (\nabla \cdot) + \eta \nabla h (\nabla h^T \cdot) & \text{(bathymetry)} \end{cases}$$

Nonlinear and time-dependent operator.

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Nonlinear and time-dependent operator.


**Goal** – adopt matrix-free methods where we avoid constructing  $\mathcal{G}$ .  
e.g., address “Difficulty” #1 (in previous slide)

Numerical work for SWE with dispersion:

[Li, Choi, Hyman 2004]; [Choi, Goulet, Jo 2011]; [Khakimzyanov Dutykh Fedotova Mitsotakis, 2020] (effectively 1d); [Patel Kumar Rajni 2020].

# SGN Constraint Form

Evolution.  $\eta_t + \nabla \cdot (\eta \mathbf{u}) = 0,$   
 $U_t + \eta \nabla \zeta + \nabla(\eta \mathbf{u} \mathbf{u}) = F(\eta, \mathbf{u}, h).$

Constraint.  $U - \mathcal{G} \mathbf{u} = 0,$   
 $\mathcal{G} \equiv \mathcal{N}_\eta + K_{\eta, h}.$  [Ignore this (for now)] 

$$F(\eta, \mathbf{u}, h) \equiv \nabla \left\{ \frac{1}{3} \eta^3 \left( \mathbf{u} \cdot \nabla (\nabla \cdot \mathbf{u}) - (\nabla \cdot \mathbf{u})^2 \right) + \frac{1}{2} \eta^2 (\mathbf{u} \cdot \nabla)^2 h + \eta^2 \nabla \cdot (\eta \mathbf{u}) \nabla \cdot \mathbf{u} + \eta \nabla \cdot (\eta \mathbf{u}) \nabla h \cdot \mathbf{u} \right\} - \nabla h \left\{ \frac{1}{2} \eta^2 \left( \mathbf{u} \cdot \nabla (\nabla \cdot \mathbf{u}) - (\nabla \cdot \mathbf{u})^2 \right) + \eta (\mathbf{u} \cdot \nabla)^2 h + \eta \nabla \cdot (\eta \mathbf{u}) (\nabla \cdot \mathbf{u}) + \nabla \cdot (\eta \mathbf{u}) (\nabla h \cdot \mathbf{u}) \right\}$$

Consider two time stepping schemes. Although unrelated, their efficacy will be related:


1) "standard" conceptual approach:

$\eta_t + \nabla \cdot (\eta \mathbf{u}) = 0,$  Explicit time-stepping  
 $U_t + \eta \nabla \zeta + \nabla(\eta \mathbf{u} \mathbf{u}) = F(\eta, \mathbf{u}, h).$  (e.g., Runge-Kutta)

$U - \mathcal{G} \mathbf{u} = 0,$  Fully implicit solve via matrix-free method (e.g., PCG).

# SGN Constraint Form

Evolution.  $\eta_t + \nabla \cdot (\eta \mathbf{u}) = 0,$   
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**Key challenge: Find a preconditioner**

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**Key challenge: Find a preconditioner**

2) Implicit-explicit (ImEx) multistep methods – avoid a fully implicit treatment of G.

$w_t = f(w, \mathbf{u}, t)$	$w = (\eta, U)^T$	Spatial discretizations (MOL)
$g(w, \mathbf{u}) = 0.$	$f = (-\nabla \cdot (\eta \mathbf{u}), -\eta \nabla \zeta - \nabla(\eta \mathbf{u} \tilde{\mathbf{u}}) + F)^T$	→ Differential algebraic equation (index-1).
	$g = U - \mathcal{N}_\eta \mathbf{u} - K_{\eta, h} \mathbf{u},$	

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$g(w, u) = 0.$	$f = (-\nabla \cdot (\eta u), -\eta \nabla \zeta - \nabla(\eta u \tilde{u}) + F)^T$	→ Differential algebraic equation (index-1).
	$g = U - \mathcal{N}_\eta u - K_{\eta,h} u,$	<b>Key challenge: Zero stability (linear implicit)</b>

---

# Previous Work: Splitting Methods

## Split into hyperbolic and elliptic subproblems

### Strang splitting

- [Bonneton, Chazel, Lannes, Marche, Tissier, 2011],
- [Samii, Dawson, 2018]
- [Lannes, March, 2015]

### Predictor-Corrector type

- [Metayer, Gavrilyuk, Hank, 2010]
- [Parisot, 2019]
- [Gavrilyuk, Shyue, 2023]
- [Noelle, Parisot, Tscherpel, 2022]

---

# Previous Work: Modify SGN

**Change the SGN equations = simplify the constraint solves.**

Asymptotically consistent block diagonal operator.

- [Lannes, Bonneton, 2009]
- [Lannes, Marche, 2015]
- [Duran, Marche, 2018]

Hyperbolic approximation

- [Favrie, Gavrilyuk, 2017]
- [Guermond, Kees, Popov, Tovar, 2019]
- [Busto, Dumbser, Escalante, Favrie, Gavrilyuk, 2021]
- [Guermond, Kees, Popov, Tovar, 2022]
- [Tkachenko, Gavrilyuk, Massoni, 2023]

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# Previous Work: Constraint Equation

## Solvers for Constraint Systems (several works are for flat bathymetry)

### Multigrid/quadrees

- [Popinet, 2015]
- [Calhoun, Berger, Private communication]

### GMRES

- [Duchene, Klein, 2022]
- [Gavrilyuk, Klein, 2022]

### Classical iteration

- [Dutykh, Clamond, Milewski, Mitsotakis, 2013]

### Existence Theory:

- [Kazerani, 2016]

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# Part II

## Preconditioner for SGN Constraint Equation

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# Preconditioning 101

Symmetric positive definite

$$Ax = f$$

Conditioning number:

$$\kappa(\mathbf{A}) \equiv \lambda_{\max}/\lambda_{\min}$$

---

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$$\mathbf{Ax} = \mathbf{f}$$

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Conjugate gradient, upper bound:

$$\|\mathbf{e}_k\|_A \leq 2 \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^k \|\mathbf{e}_0\|_A, \quad \|\mathbf{x}\|_A \equiv \mathbf{x}^T \mathbf{Ax}.$$

---

# Preconditioning 101

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Preconditioned:

$$\mathbf{P}^{-1/2} \mathbf{A} \mathbf{P}^{-1/2} \mathbf{y} = \mathbf{P}^{-1/2} f$$

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# Preconditioner 601

Goal: Solve many SGN constraints of form:

$$\mathcal{G}_{\eta,h} \mathbf{u} = \mathbf{f}$$

---

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Choice:

1. Easy to solve linear systems

$$\mathcal{A} = \sigma \mathbf{I} - \alpha \nabla \nabla.$$

2. Operator  $\mathcal{A}^{-1} \mathcal{G}_{\eta,h}$  is close to identity

---

# Preconditioner 601

Goal: Solve many SGN constraints of form:

$$\mathcal{G}_{\eta,h} \mathbf{u} = \mathbf{f}$$

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1. Easy to solve linear systems

$$\mathcal{A} = \sigma \mathbf{I} - \alpha \nabla \nabla.$$

2. Operator  $\mathcal{A}^{-1} \mathcal{G}_{\eta,h}$  is close to identity

Minimize (proxy):  $\kappa_{\mathcal{A}}(\mathcal{G}_{\eta,h}) \equiv \frac{\mu_{\max}}{\mu_{\min}}$

---

# Choice of A: Mathematical preliminaries

Operator  $\mathcal{G} \equiv \mathcal{N}_\eta + \mathcal{K}_{\eta,h}$  consists of two contributions:

1)  $\mathcal{N}_\eta \equiv \eta \mathbf{I} - \nabla \frac{1}{3} \eta^3 \nabla \cdot$

2)  $\mathcal{K}_{\eta,h} \equiv -\nabla \left( \frac{1}{2} \eta^2 \nabla h^T \cdot \right) + \frac{1}{2} \eta^2 \nabla h (\nabla \cdot) + \eta \nabla h (\nabla h^T \cdot)$

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Variational forms defined on the Hilbert space:

$$\|\mathbf{u}\|_{\text{div}}^2 := \int_{\Omega} \|\mathbf{u}\|^2 + (\nabla \cdot \mathbf{u})^2 \, dx, \quad H_{\text{div}}(\Omega) := \{\mathbf{u} \mid \|\mathbf{u}\|_{\text{div}} < \infty\}$$

Boundary conditions  $(\mathbf{n} \cdot \mathbf{u} = 0 \text{ on } \partial\Omega)$  for non-periodic case

---

# Variational Forms

$A : H_{\text{div}}(\Omega) \times H_{\text{div}}(\Omega) \rightarrow \mathbb{R}$ , and  $B : H_{\text{div}}(\Omega) \times H_{\text{div}}(\Omega) \rightarrow \mathbb{R}$

$$A[\mathbf{u}, \mathbf{v}] = (\mathbf{u}, \mathcal{A}\mathbf{v}) = \int_{\Omega} (\mathbf{u}^T \quad \nabla \cdot \mathbf{u}) \begin{pmatrix} \sigma^* \mathbf{I} & 0 \\ 0 & \alpha^* \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \nabla \cdot \mathbf{v} \end{pmatrix} d\mathbf{x}$$

---

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$$B[\mathbf{u}, \mathbf{v}] = (\mathbf{u}, \mathcal{G}_{\eta, h}\mathbf{v}) = \int_{\Omega} (\mathbf{u}^T \quad \nabla \cdot \mathbf{u}) \mathbf{M}(\mathbf{x}) \begin{pmatrix} \mathbf{v} \\ \nabla \cdot \mathbf{v} \end{pmatrix} dx$$

$$\mathbf{M}(\mathbf{x}) = \eta(\mathbf{x}) \begin{pmatrix} \mathbf{I} + \nabla h(\mathbf{x}) \nabla h^T(\mathbf{x}) & \frac{1}{2} \eta(\mathbf{x}) \nabla h(\mathbf{x}) \\ \frac{1}{2} \eta(\mathbf{x}) \nabla h(\mathbf{x})^T & \frac{1}{3} \eta^2(\mathbf{x}) \end{pmatrix}$$

---

# Preconditioner Choice

Love to pick coefficients of

$$\mathcal{A} = \sigma \mathbf{I} - \alpha \nabla \nabla.$$

To minimize:

$$\kappa_{\mathcal{A}}(\mathcal{G}_{\eta,h}) \equiv \frac{\mu_{\max}}{\mu_{\min}}$$

$$\mu_{\min} = \inf_{0 \neq \mathbf{u} \in H_{\text{div}}(\Omega)} \frac{B[\mathbf{u}, \mathbf{u}]}{A[\mathbf{u}, \mathbf{u}]} \quad \text{and} \quad \mu_{\max} = \sup_{0 \neq \mathbf{u} \in H_{\text{div}}(\Omega)} \frac{B[\mathbf{u}, \mathbf{u}]}{A[\mathbf{u}, \mathbf{u}]}$$

---

# Preconditioner: Main Result

Don't quite get there, but we are close.

$$A\mathbf{u} = \sigma^* \mathbf{u} - \alpha^* \nabla(\nabla \cdot \mathbf{u})$$

For variable bathymetry, let:

$$\sigma^* \equiv \max_{\mathbf{x} \in \Omega} \left( \eta(\mathbf{x}) + \frac{1}{6}(4 + \sqrt{13})\eta(\mathbf{x})|\nabla h(\mathbf{x})|^2 \right)$$

$$\alpha^* \equiv \frac{1}{6}(4 + \sqrt{13}) \eta_{\max}^3$$

$$\kappa_{\text{ub}} \equiv \max \left\{ \frac{\sigma^*}{\eta_{\min}}, \left( \frac{4 + \sqrt{13}}{4 - \sqrt{13}} \right) \left( \frac{\eta_{\max}}{\eta_{\min}} \right)^3 \right\}$$

---

# Preconditioner: Main Result

Theorem: [Feng, DS, Choi, '26]

1. The conditioning number is bounded

$$\kappa_{\mathcal{A}}(\mathcal{G}_{\eta,h}) \leq \kappa_{\text{ub}}$$

---

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$$(\kappa_{\text{ub}})^{-1} \leq \lambda \leq 1 \quad \text{where} \quad \mathcal{G}_{\eta,h} \mathbf{u} = \lambda \mathcal{A} \mathbf{u}$$

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3. The preconditioner is quasioptimal.

---

# Quasioptimal in the sense:

(P) Minimize:  $\gamma$

Subject to:  $\mathbf{M}(\mathbf{x}) \preceq \begin{pmatrix} \sigma \mathbf{I} & 0 \\ 0 & \alpha \end{pmatrix} \preceq \gamma \mathbf{M}(\mathbf{x})$  holds for  $\mathbf{x} \in \Omega$ ,

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## Theorem

*For any functions  $\eta, \nabla h \in L^\infty(\Omega)$  with  $\eta_{\min} > 0$ , the bound  $\kappa_{\text{ub}}$  satisfies*

$$\gamma_{\text{P}}^* \leq \kappa_{\text{ub}} \leq \left( \frac{4 + \sqrt{13}}{4 - \sqrt{13}} \right) \gamma_{\text{P}}^*.$$

*In the case of a flat bathymetry ( $\nabla h = 0$ ) then  $\kappa_{\text{ub}} = \gamma_{\text{P}}^*$ .*

---

# Remarks:

- Operator bound = mesh independent convergence rates.

See also mesh independent bounds

- R. C. Kirby, *From functional analysis to iterative methods*, SIAM Review, 52, (2010), 269-293.
- M. Gilles, A. Townsend, *Continuous analogues of Krylov subspace methods for differential operators*, 57 SINUM, (2019), 237-246.

---

# Remarks:

- Operator bound = mesh independent convergence rates.
- Quasioptimal implies the scaling cannot be overcome

$$\text{(Large contrast ratio)} \quad \kappa_{\text{ub}} \sim (\eta_{\text{max}}/\eta_{\text{min}})^3$$

$$\text{(Large variations in bathymetry)} \quad \kappa_{\text{ub}} \sim \eta_{\text{max}} |\nabla h|_{\text{max}}^2$$

See also mesh independent bounds

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---

# 1D PCG Test

Test case PDE:

$$\mathcal{G}_{\eta,h}u = b \quad \text{where} \quad b(x) = \cos(4\pi x)$$

Where

$$\begin{aligned} \mathcal{G}_{\eta,h}\mathbf{u} \equiv & \eta\mathbf{u} - \nabla \left( \frac{1}{3}\eta^3 \nabla \cdot \mathbf{u} \right) - \nabla \left( \frac{1}{2}\eta^2 \nabla h \cdot \mathbf{u} \right) \\ & + \frac{1}{2}\eta^2 \nabla h (\nabla \cdot \mathbf{u}) + \eta \nabla h (\nabla h \cdot \mathbf{u}) \end{aligned}$$

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Test two sources of problem difficulty (conditioning).

$$\begin{aligned} h(x) &= 1 + h_0 \exp \left( -(x - 1/2)^2 / \sigma^2 \right), \quad \sigma = 1/20 \\ \eta(x) &= 1 + \eta_0 \cos(4\pi x)^2. \end{aligned}$$

Problem completely determined by two parameters.

---

# 1D PCG: Two Sources of Problem Difficulty

Water depth contrast, and bathymetry gradient scales as:

$$\frac{\eta_{\max}}{\eta_{\min}} = \eta_0 + 1 \quad \text{and} \quad |h_x|_{\max} \approx 17.2|h_0|.$$

---

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Source 1: High contrast problems  $\eta_{\max}/\eta_{\min} \gg 1$

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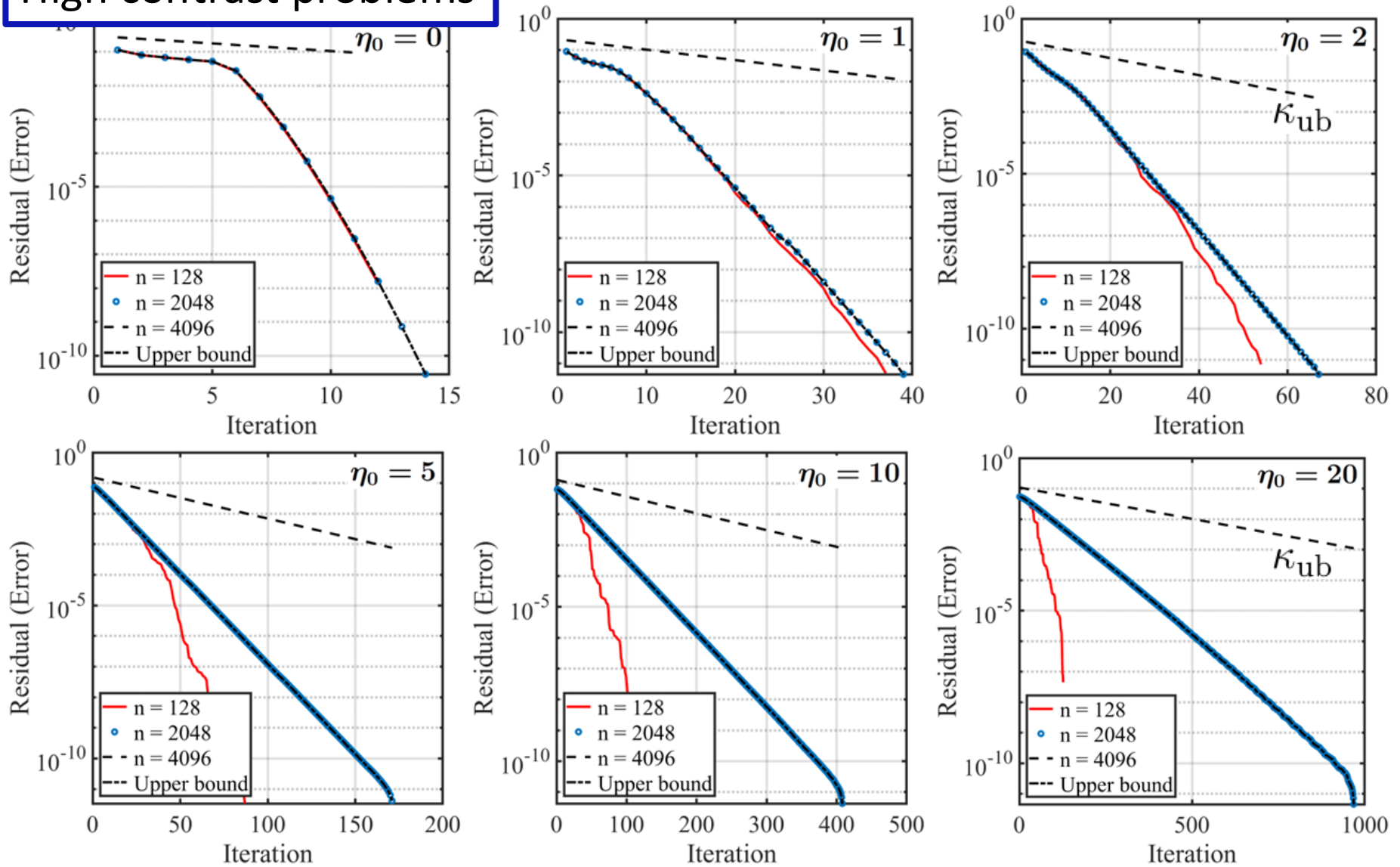
Source 1: High contrast problems  $\eta_{\max}/\eta_{\min} \gg 1$

$$\kappa_{\text{ub}} \sim (\eta_0 + 1)^3$$

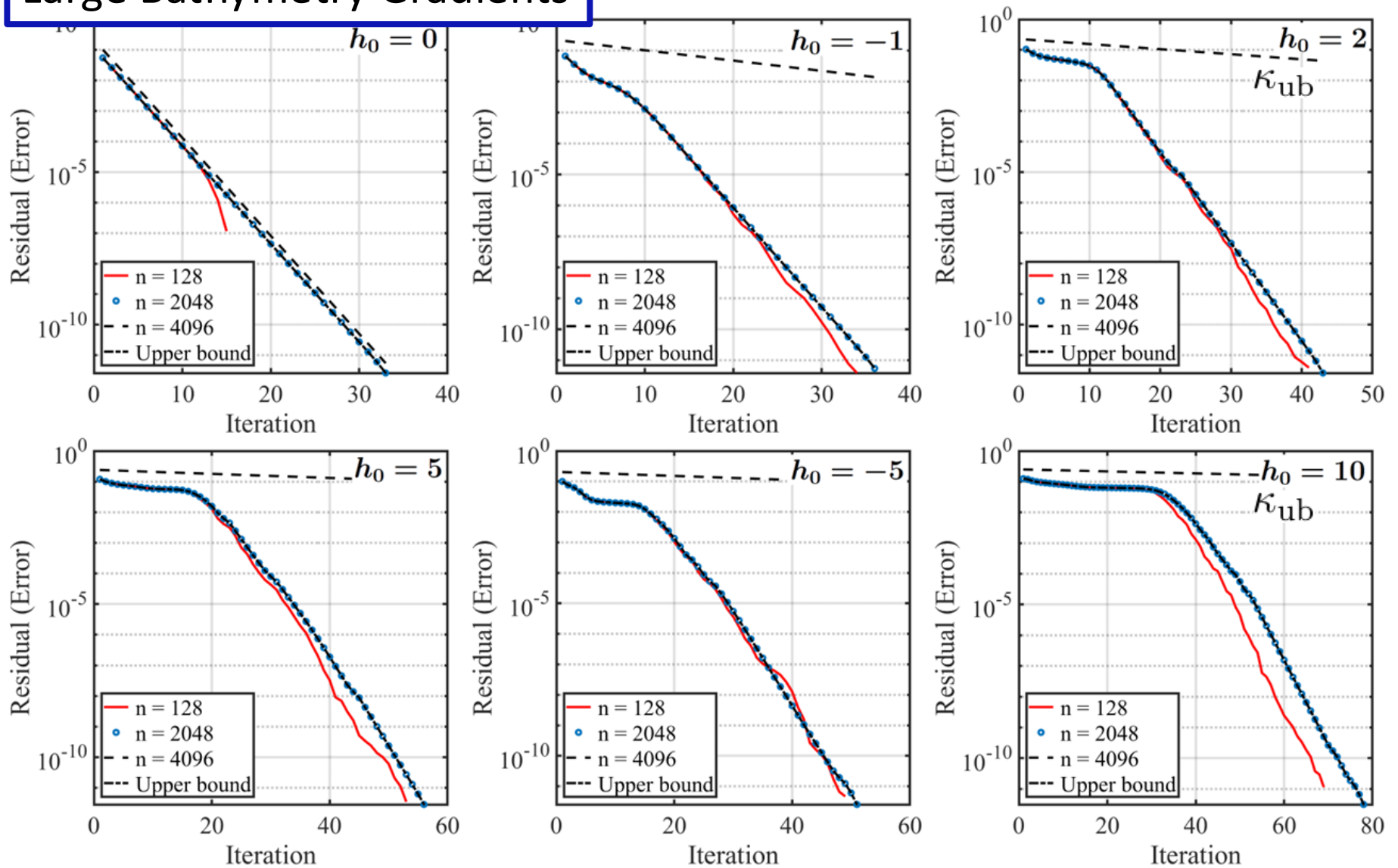
Source 2: Large bathymetry gradients  $\max_{\mathbf{x} \in \Omega} \eta(\mathbf{x}) |\nabla h(\mathbf{x})|^2 \gg 1$

$$\kappa_{\text{ub}} \sim (17.2)^2 \eta_0 h_0^2$$

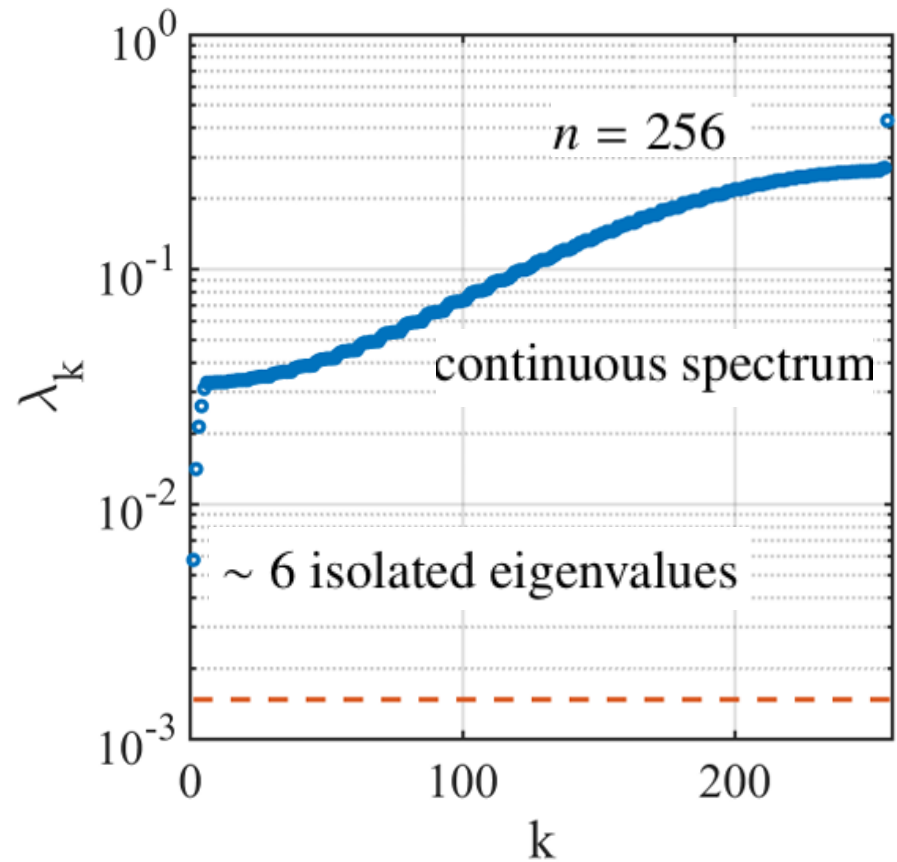
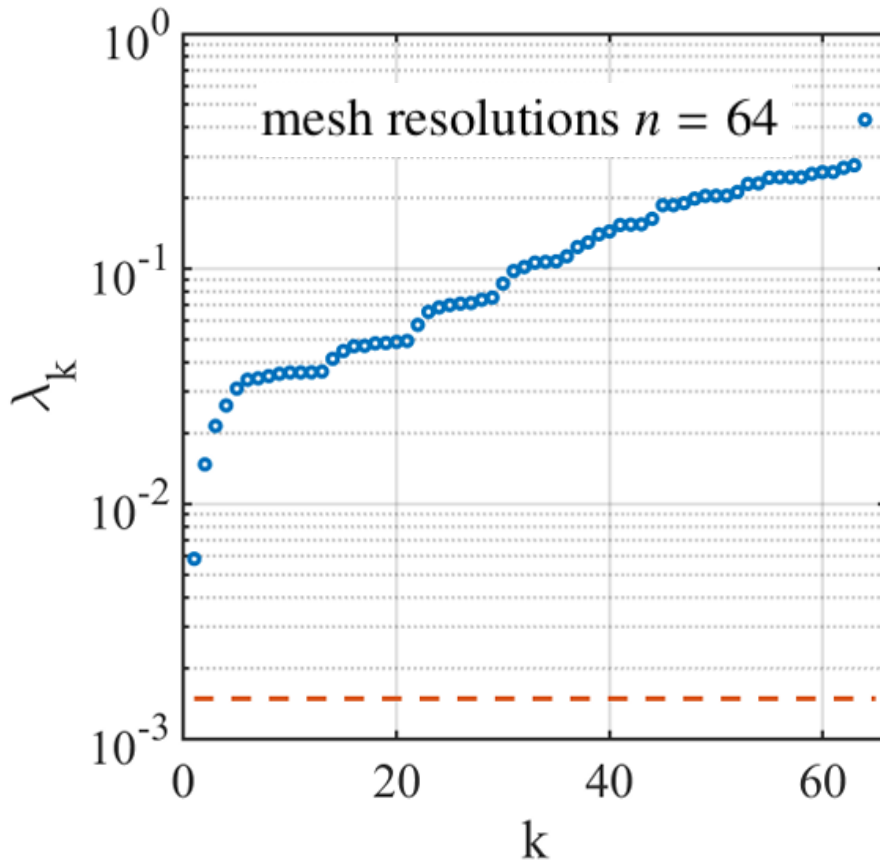
# High contrast problems



# Large Bathymetry Gradients



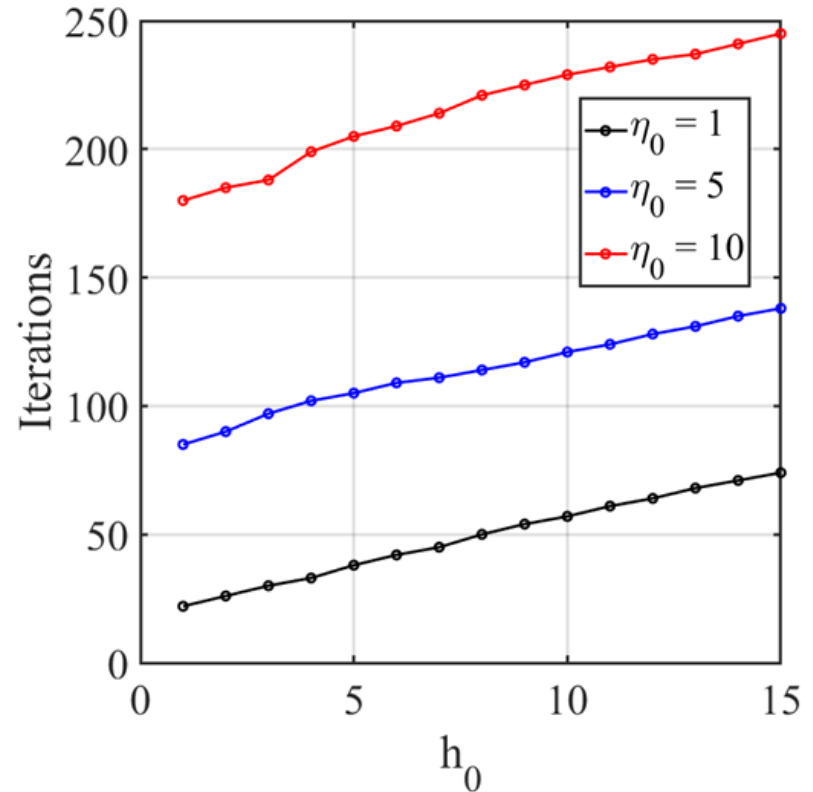
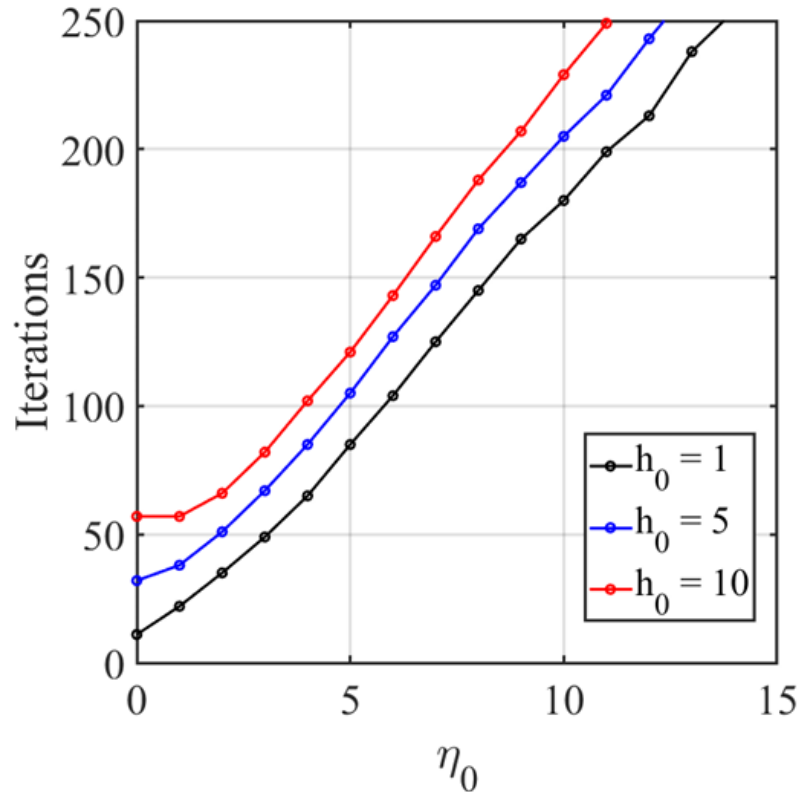
# Generalized Eigenvalues



Plot of  $k$ th eigenvalue versus  $k$  for:  $\eta_0 = h_0 = 1$

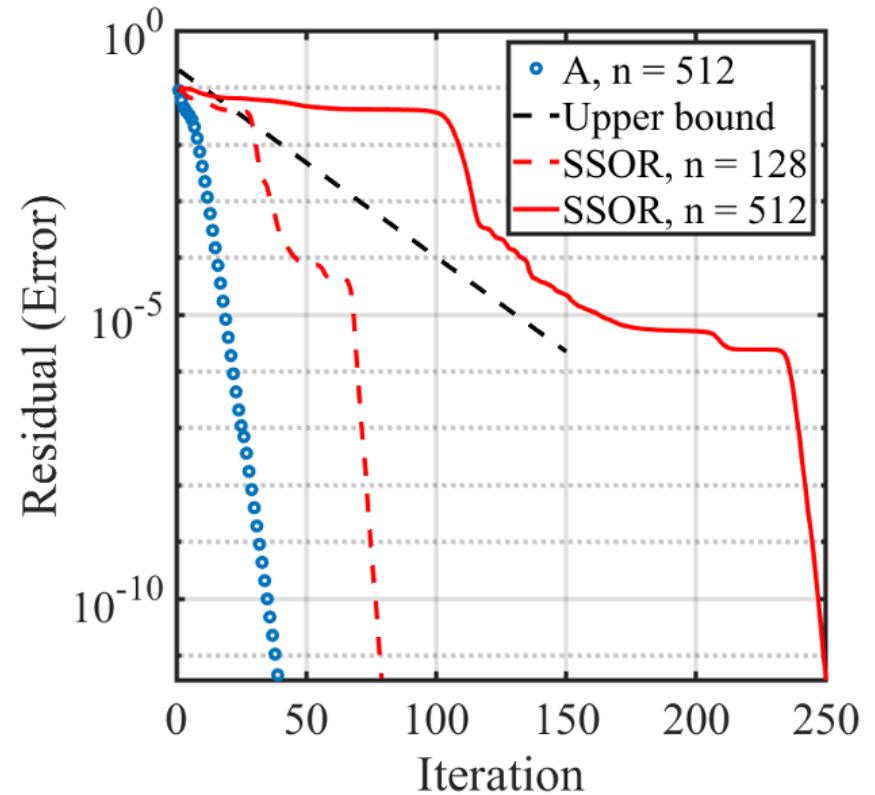
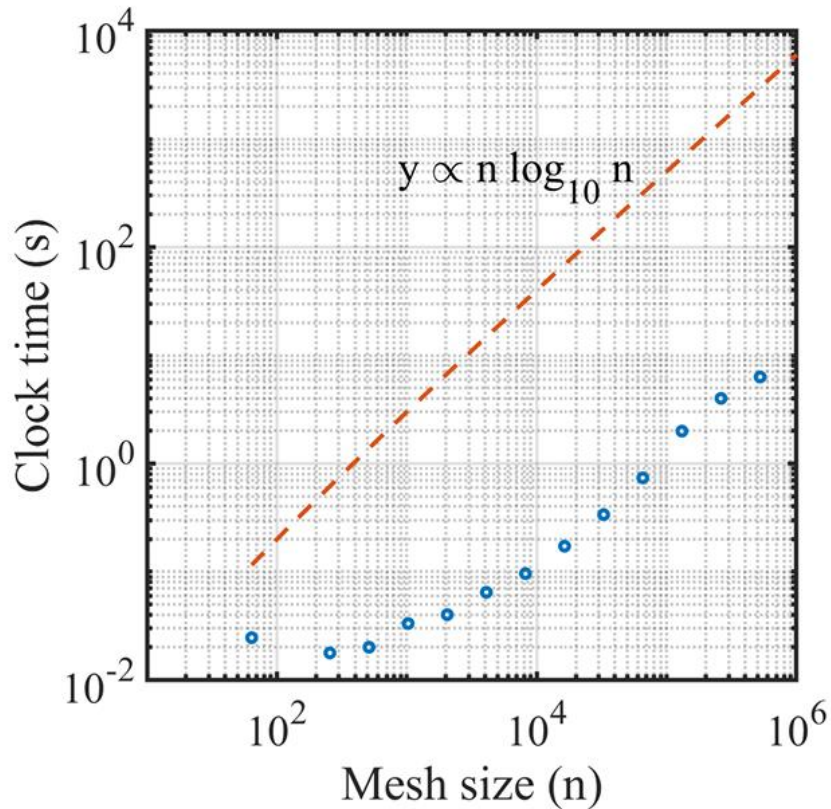
Provides an **a Posteriori PCG Convergence Bound with "Knee"**

# 1D PCG: Iterations vs Problem Difficulty



Solution to residual  $\varepsilon < 10^{-6}$  (PCG norm); mesh size 512.

# 1D PCG: Clock Time and SSOR Comparison



Clock time to solve  $\varepsilon < 10^{-12}$   
 $\eta_0 = h_0 = 1$

Comparison with SSOR.

---

# 2D PCG Test

Manufactured Solution:

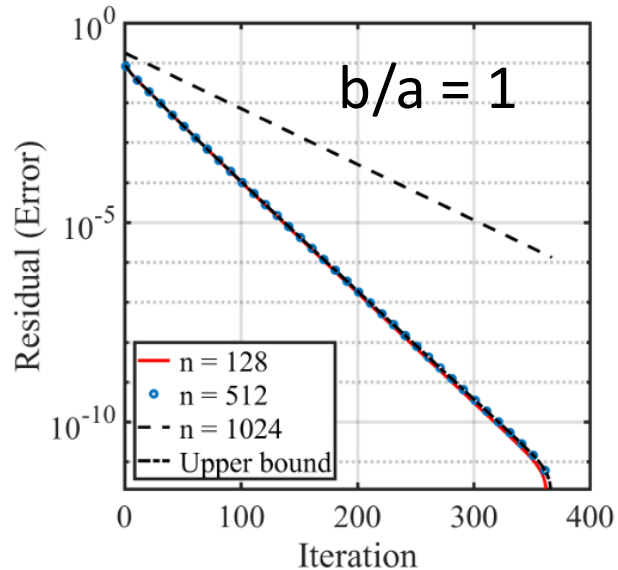
$$U = (\cos(4\pi x), \cos(4\pi y))$$

$$h(x, y) = \begin{cases} 1 - \frac{1}{2} \cos^2\left(\frac{\pi r}{2r_0}\right), & |r| \leq r_0 \\ 1, & |r| \geq r_0 \end{cases}, \quad \text{where } r = \sqrt{a^2 x^2 + b^2 y^2}, \quad r_0 = \frac{1}{2}$$

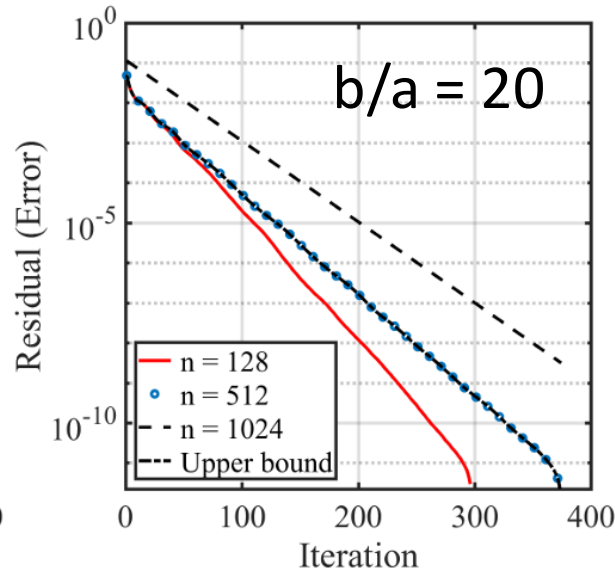
$$\eta(x, y) = h(x, y) + \exp(\cos(2\pi x)) + \frac{1}{4} \sin(4\pi y).$$

Test 2d effects on bathymetry gradient: eccentricity of ellipse.

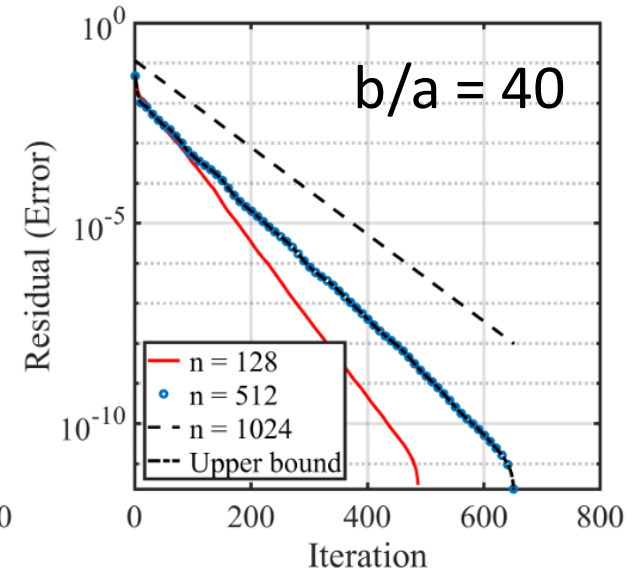
# 2D Preconditioned Conjugate Gradient



Eccentricity Low



High



Very high

Eccentricity – no significant effect other than large bathymetry gradients.

---

# Part III

## Semi-Implicit Time Integration for SGN (Stability).

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# ImEx linear multistep for index-1 DAEs

Theoretical tool to formulate schemes for DAEs (e.g., [Hairer Wanner, Vol. II], see also [Constantinescu, Sandu 2010]; ImEx LMMs [Crouzeix, 1980], [Ascher, Ruuth, Wetton, 1995]):

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- Where  $\mathcal{G} = \mathcal{A} + \mathcal{B}$ , Choice is NOT unique!
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What we want is to take  $\mathcal{A}$  linear and constant coefficient

# ImEx linear multistep for index-1 DAEs

Linear multistep (after taking  $\epsilon \rightarrow 0$ )

ImEx scheme

(ImEX DAE)

$$\frac{1}{\Delta t} \sum_{j=0}^s a_j \mathbf{w}^{n+j} = \sum_{j=0}^s b_j \mathbf{f}(\mathbf{w}^{n+j}, \mathbf{u}^{n+j}, t^{n+j}),$$

$$0 = \sum_{j=0}^s c_j (\mathbf{U}^{n+j} - \mathbf{A} \mathbf{u}^{n+j}) - b_j \mathbf{B} \mathbf{u}^{n+j}. \quad \text{Where } \mathcal{G} = \mathcal{A} + \mathcal{B},$$

Example: Semi-implicit  
backward differentiation 3

$$c_2 = 1, c_1 = 0, c_0 = 0$$

$$a_2 = 3/2, a_1 = -2, a_0 = 1/2$$

$$b_2 = 0, b_1 = 2, b_0 = -1$$

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So what can go wrong?

Answer: Zero-stability, e.g., stability with  $\Delta t = 0$

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Require: solutions  $\mathbf{u}_n$  uniformly bounded for all  $n$ .

- Not a trivial property due to the explicit stiff terms
- Property depends on **BOTH** coefficients **AND** splitting.

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# Zero-stability criteria

Simplified case: Assume  $\mathcal{G}$  is time-independent (but can vary in space)

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Yields the following zero-stability criteria

Define:  $\mathcal{D} := \{\mu \in \mathbb{C} : c(z) + (\mu - 1)b(z) \text{ has stable roots}\}$

$$c(z) := z^r + c_{r-1}z^{r-1} + \dots + c_0$$

$$b(z) := b_{r-1}z^{r-1} + \dots + b_0.$$

Then (\*) yields stable dynamics if:  $\text{eig}(\mathcal{A}^{-1}\mathcal{G}) \in \mathcal{D}$

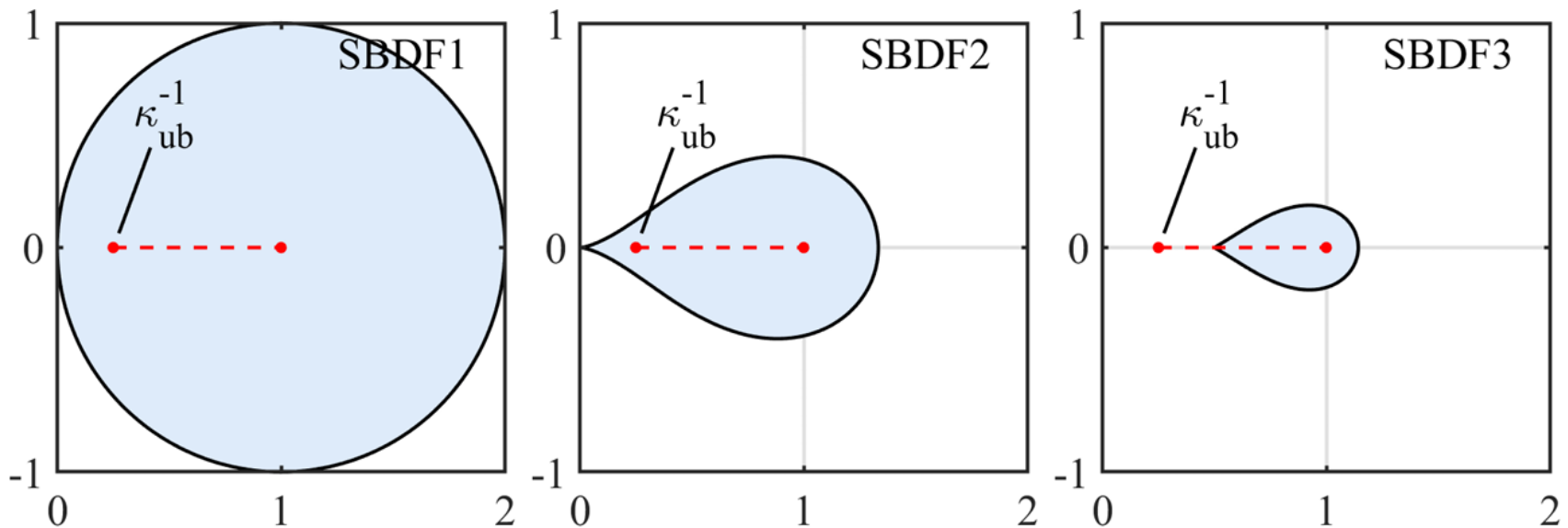
# Significance of the result

Combined with eigenvalue bound (main result):

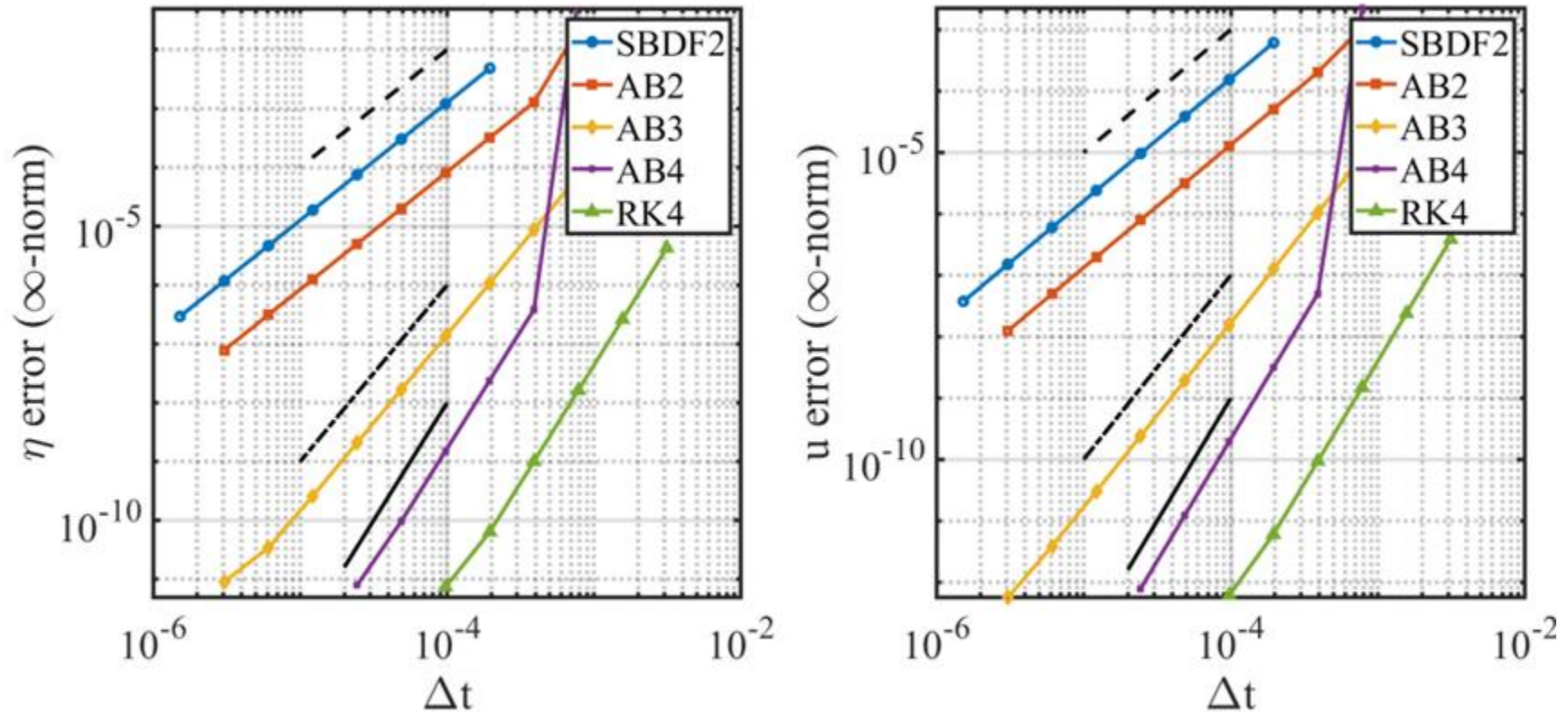
$$(\kappa_{\text{ub}})^{-1} \leq \lambda \leq 1$$

Stability condition:

$$(\kappa_{\text{ub}})^{-1} \in \mathcal{D}$$

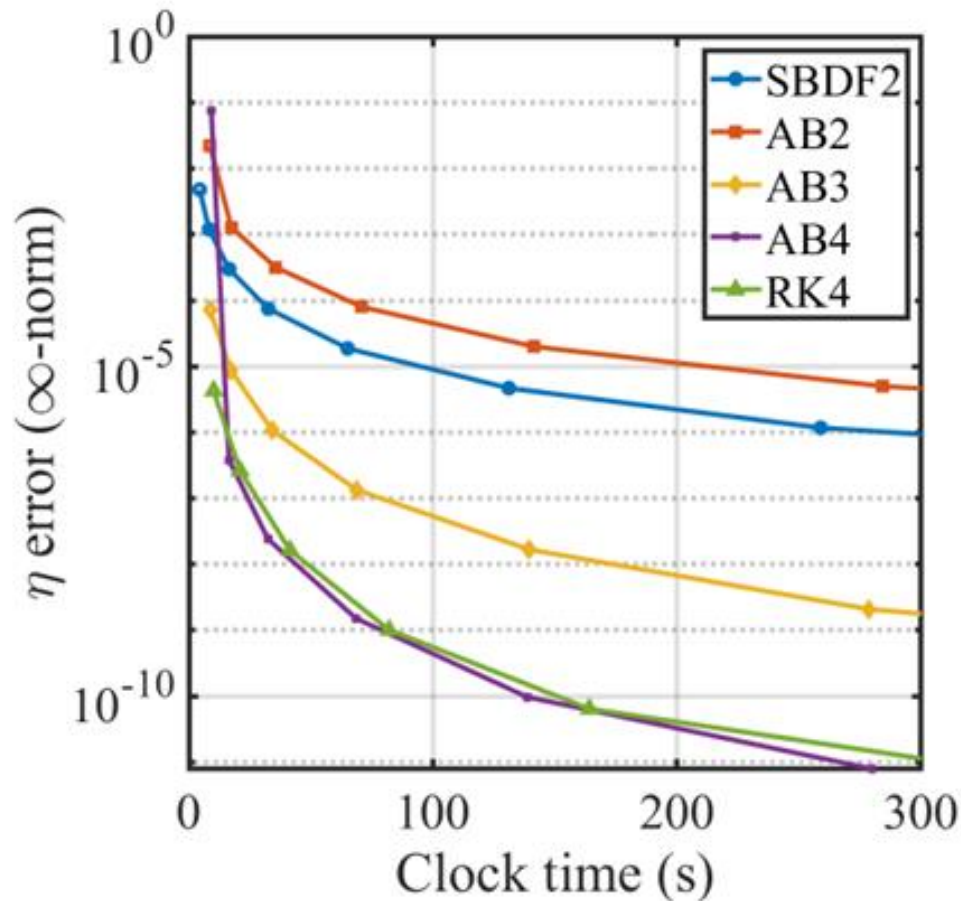


# 1d SGN: Convergence Study



Manufactured solution (256 mesh;  $T = 1$ )

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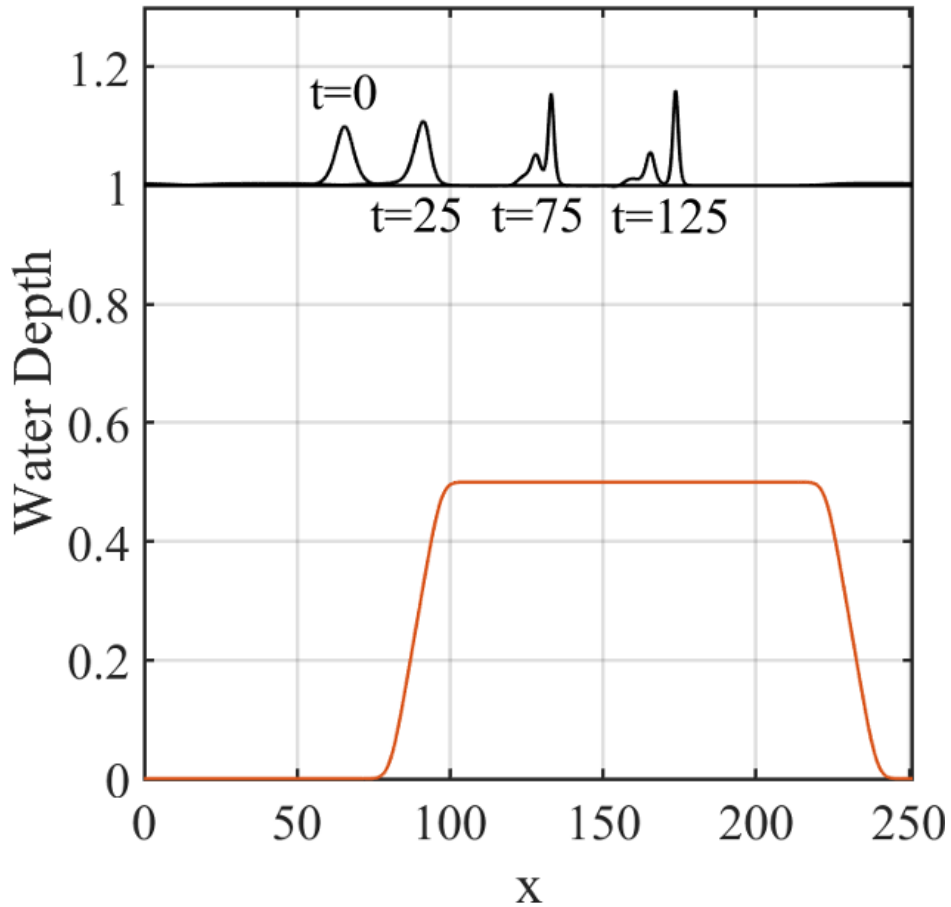
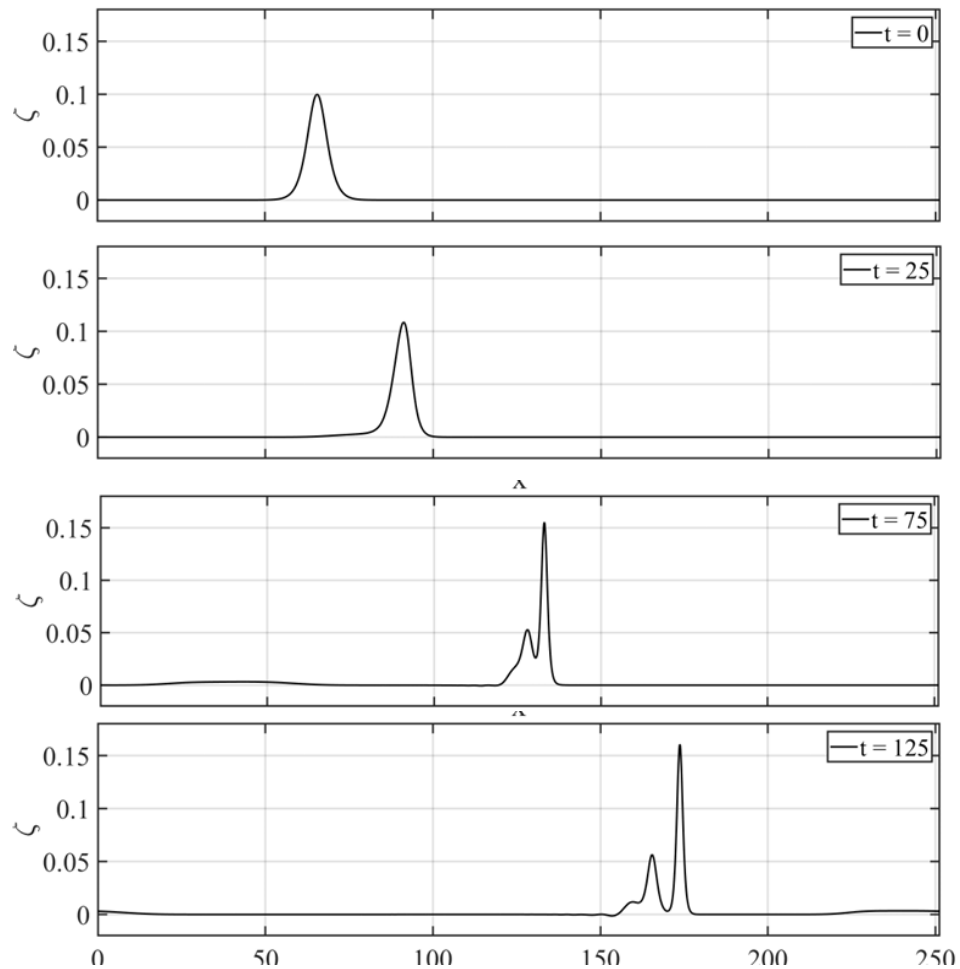


- Manufactured solution (256 mesh;  $T = 1$ ).

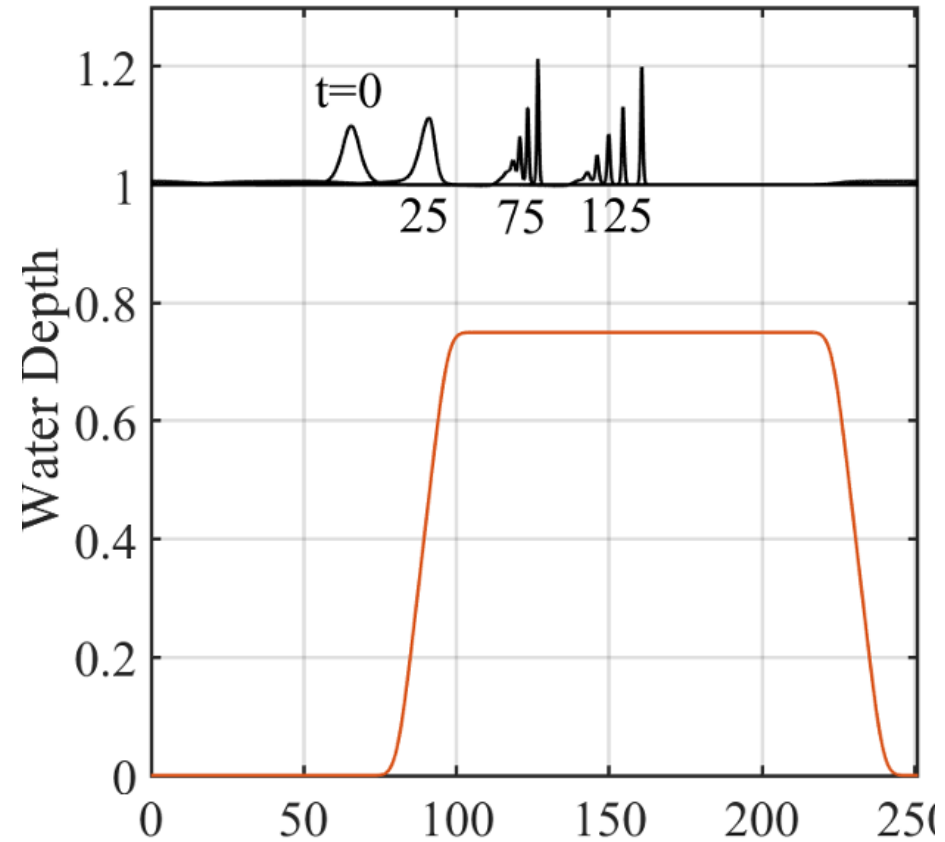
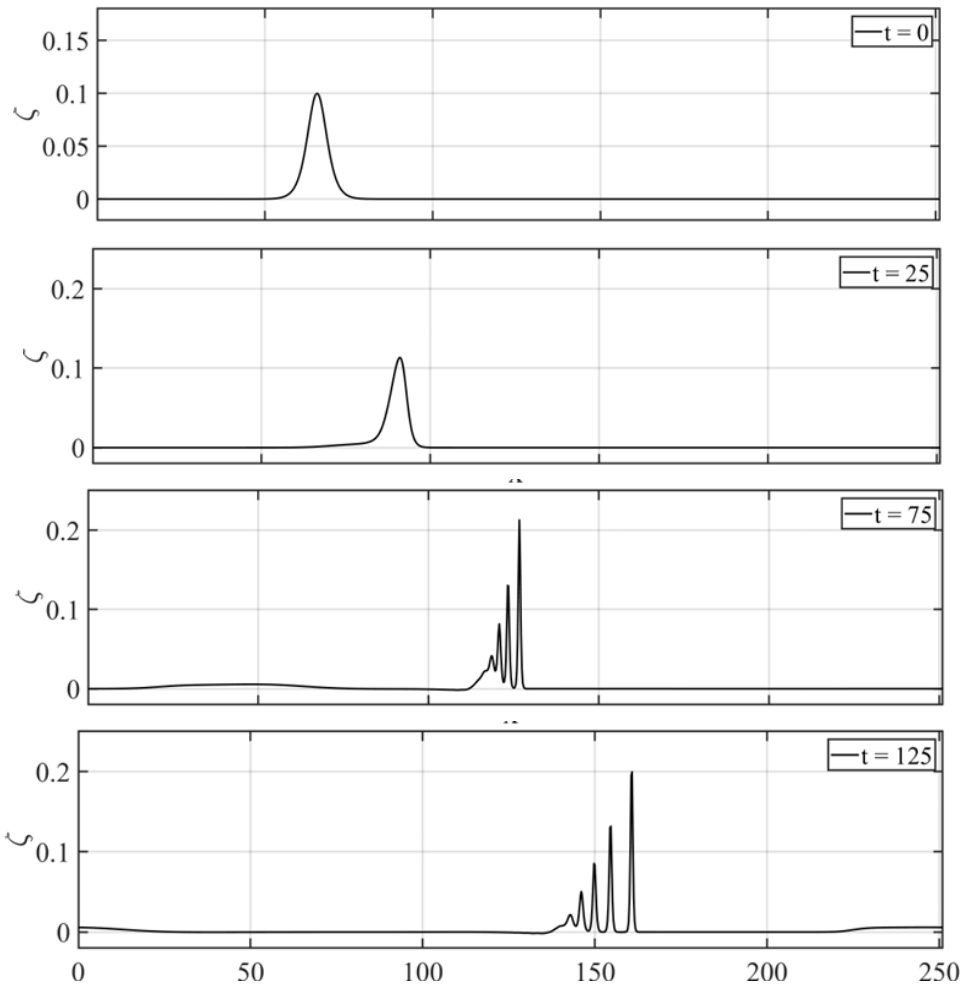
Lenovo ThinkPad P14s Gen2 laptop  
MATLAB (Version R2023b)

Processor: 11th Gen Intel(R) Core(TM) i7-1165G7 @ 2.80 GHz; RAM: 32.0 GB; 64-bit operating system.

# Solitary Wave on a Shelf



# Solitary Wave on a Shelf



# 2D SGN: Solitary Wave Over Bump

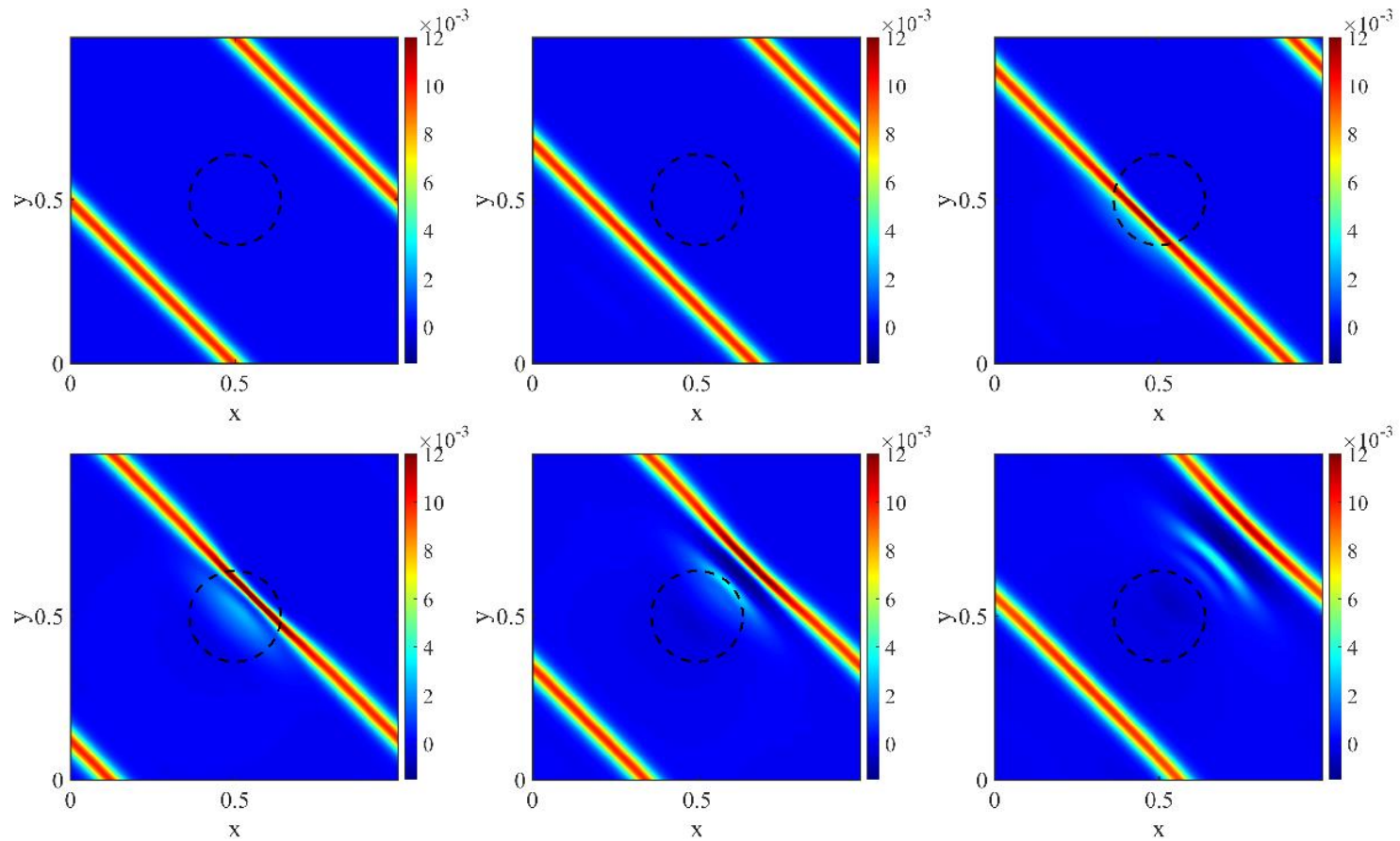


Fig. 13: Contour plots of  $\zeta$  showing a solitary wave passing over an underwater circular 2d barrier (6.18). The dashed line shows the level surface  $b(x,y) = \frac{1}{2}0.75a$ . Snapshots (top-bottom, l-r) correspond to  $t = 0, .8, 1.6, 2.4, 3.2, 4.0$ .

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# Summary and Discussion

**#1.** SGN models incorporate dispersive effects beyond SWE.

**#2.** Result: new preconditioners coefficients with mesh independent a priori bounds on  $\kappa$ . This enables use of fast solvers that leverage constant coefficient operators (spectral methods, boundary integral).

**#3.** Operator splitting for DAEs outperform other methods; stability is subtle.

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# Summary and Discussion

**#1.** SGN models incorporate dispersive effects beyond SWE.

Which model in the fluid hierarchy will supersede SWE? What role will HPC play in picking winners?

**#2.** Result: new preconditioners coefficients with mesh independent a priori bounds on  $\kappa$ . This enables use of fast solvers that leverage constant coefficient operators (spectral methods, boundary integral).

Should we solve exactly fluid models, or approximations? (e.g., hyperbolic approximations; models that approximate linear operators).

**#3.** Operator splitting for DAEs outperform other methods; stability is subtle.

Where do practitioners land in trade-off: Robust codes that are less efficient VS efficient codes that require some expertise (e.g., code fails when expert leaves room).

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# Thank You!

- 1) L. Feng, DS, W. Choi, Preconditioning and linearly implicit time integration for the Serre-Green-Naghdi Equations, JCP, 558:1 (2026), 114849.
- 2) B. Seibold, DS, D. Zhou, Unconditional stability for multistep IMEX schemes: Practice, JCP, 376:1 (2019), 295-321.
- 3) R. R. Rosales, B. Seibold, DS, D. Zhou, Unconditional stability for multistep IMEX schemes: Theory, SINUM, 55:5 (2017), 2336-2360.

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# Choice of A: Use variational techniques

Obtain a bound:  $\gamma(u, Au) \leq (u, Gu) \leq (u, Au)$

With implicit operator/preconditioner:  $A = \sigma I - \alpha \nabla \nabla$ .

With coefficients:  $\alpha = \frac{4}{3} \eta_{\max}^3$        $\sigma = \eta_{\max} \left( 1 + \frac{4}{3} (\nabla h)_{\max}^2 \right)$

---

“Estimated” quantities:  $\eta_{\max} := \max_{\mathbf{x} \in \Omega} \eta(\mathbf{x})$        $\eta_{\min} := \min_{\mathbf{x} \in \Omega} \eta(\mathbf{x})$ ,       $(\nabla h)_{\max} = \max_{\mathbf{x} \in \Omega} \|\nabla h(\mathbf{x})\|$

Formula:  $\gamma = \min \left\{ \left( \frac{\eta_{\min}}{\eta_{\max}} \right)^3, \left( \frac{\eta_{\min}}{\eta_{\max}} \right) \left( 1 + \frac{4}{3} (\nabla h)_{\max}^2 \right)^{-1} \right\}$

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# Choice of A: Use variational techniques

Introduce new variables:  $\phi := \nabla \cdot \mathbf{u}$  and  $\mathbf{w}(\mathbf{x}) = \nabla h(\mathbf{x})$

$$(\mathbf{u}, \mathcal{G}\mathbf{u}) = \int_{\Omega} \underbrace{\eta \|\mathbf{u}\|^2 + \frac{1}{3} \eta^3 \phi^2}_{(\mathbf{u}, \mathcal{N}_{\eta} \mathbf{u})} + \underbrace{\eta (\mathbf{w}^T \mathbf{u})^2 + \eta^2 (\mathbf{w}^T \mathbf{u}) \phi}_{(\mathbf{u}, \mathcal{K}_{\eta, h} \mathbf{u})} \, d\mathbf{x}$$

---

Key observation (look for a sums of squares):

$$(\mathbf{u}, \mathcal{G}\mathbf{u}) = \int_{\Omega} \eta |\mathbf{u}|^2 + \mathbf{a}^T \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \mathbf{a} \, d\mathbf{x}, \quad \text{where } \mathbf{a}(\mathbf{x}) = \begin{pmatrix} \eta^{1/2} \mathbf{w} \\ \eta^{3/2} \phi \end{pmatrix}$$

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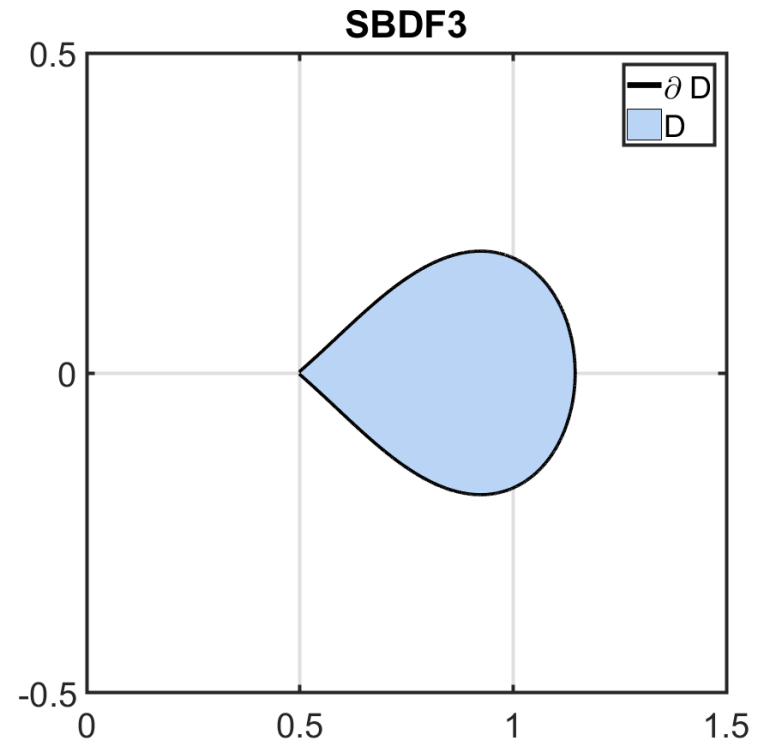
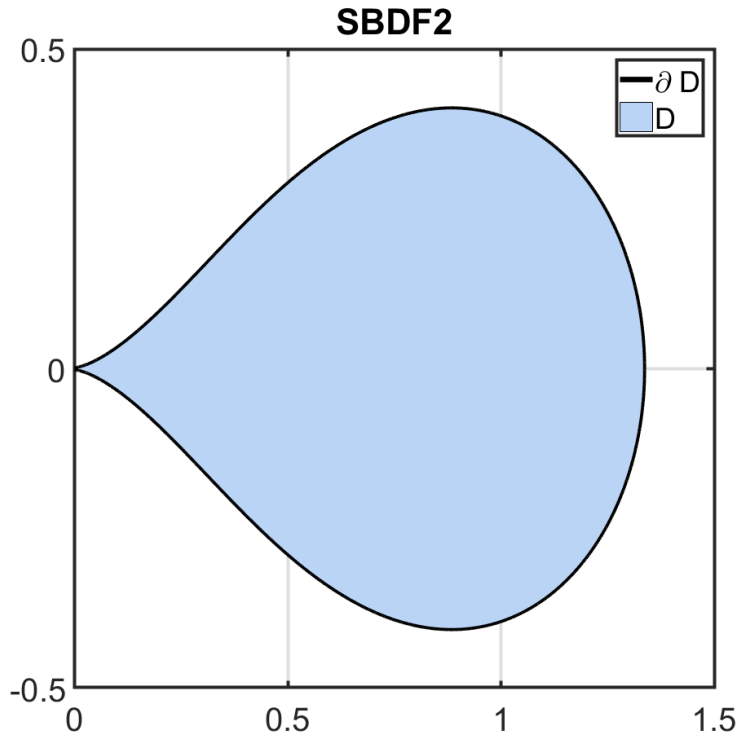
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# Zero-stability criteria

Need generalized eigenvalues  $\mathcal{G}u = \lambda Au \rightarrow \text{eig}(\mathcal{A}^{-1}\mathcal{G}) \in \mathcal{D}$



You need the eigenvalues clustered near 1  $\rightarrow$  Just like preconditioning

Caveat: Need bounds (bounds become fundamentally worse/impossible at 3<sup>rd</sup> order without simultaneously choosing time-stepping schemes!)