The solution of nonlinear optimal control problems in motion planning by non-AI-based methods

Jörg Wensch

Go20 Malta May 2025

HTWD Hochschule für Technik und Wirtschaft Dresden University of Applied Sciences

Overview

- Optimal control: the setting
- Linear optimal control problems
  - Analytical solution
  - Numerical solution
- Nonlinear problems - Numerical solution
- Controlling motion
- Something (not so) different...

Optimal control problems

Generall setting

We consider a differential equation (ODE) with boundary conditions (BC) where an objective function (OF) has to be minimized with respect to the control variable $\mathbf{u}$:

$$\begin{aligned} \text{ODE:} \quad & \mathbf{y}' = f(t, \mathbf{y}(t), \mathbf{u}(t)), \mathbf{u} \in R(t, \mathbf{y}(t)) \\ \text{BC:} \quad & 0 = B(\mathbf{y}(0), \mathbf{y}(T)) \\ \text{OF:} \quad & \phi(\mathbf{y}, \mathbf{u}) = \int_0^T g(t, \mathbf{y}, \mathbf{y}) \, \mathrm{d}t + C(\mathbf{y}(0), \mathbf{y}(T)) \to \min_{\mathbf{u}} \end{aligned}$$

HTW Hochschule für Technik und Wirtschaft Dresden University of Applied Sciences

Two Approaches

Discretize first, then optimize

- The differential equation is discretized by simple symmetric finite differences or more accurate methods like RK methods.
- The objective function is discretized by quadrature.
- This forms a huge constrained optimization problem, to be solved by a professional optimizer. Constraints come from the ODE and BC, objective function is discretized OF.

Simplest symmetric version (Gaussian Kollocation)

$$\text{ODE:} \qquad \frac{\mathbf{y}_{n+1} - \mathbf{y}_n}{h} = f\left(t_{n+1/2}, \frac{\mathbf{y}_n + \mathbf{y}_{n+1}}{2}, \mathbf{u}_{n+1/2}\right)$$

$$\text{OF:} \qquad \phi(\mathbf{y}, \mathbf{u}) = h \sum_{n=0}^{N-1} g\left(_{n+1/2}, \frac{\mathbf{y}_n + \mathbf{y}_{n+1}}{2}, \mathbf{u}_{n+1/2}\right) \mathrm{d}t + \dots$$

Two Approaches

Optimize first, then discretize

- We couple the constraints (ODE) to the objective function by Lagrangian multiplyers

$$\Phi(t) = \int_0^T \left[g(t, \mathbf{y}, \mathbf{y}) + \lambda^T (\mathbf{y}' - f(t, \mathbf{y}, \mathbf{u})) \right] \, \mathrm{d}t + \dots$$

- Variational calculus leads to the Pontryagin Maximum Principle

$$\begin{aligned} \mathbf{y}' &= f(t, \mathbf{y}, \mathbf{u}) \\ \lambda' &= -\frac{\partial f}{\partial \mathbf{y}}^T \lambda - \frac{\partial g}{\partial \mathbf{y}}^T \\ \mathbf{u} &= \operatorname{argmin}_{\mathbf{u}} H(t, \mathbf{y}, \lambda, \mathbf{u}) \end{aligned}$$

with the Hamiltonian $H = g(t, \mathbf{y}, \mathbf{u}) + \lambda^T f(t, \mathbf{y}, \mathbf{u})$.

Two Approaches

Optimize first, then discretize

Note that Pontryagin Maximum Principle

$$\mathbf{y}' = f(t, \mathbf{y}, \mathbf{u})$$
$$\lambda' = -\frac{\partial f}{\partial \mathbf{y}}^T \lambda - \frac{\partial g}{\partial \mathbf{y}}^T$$
$$\mathbf{u} = \operatorname{argmin}_{\mathbf{u}} H(t, \mathbf{y}, \lambda, \mathbf{u})$$

with the Hamiltonian $H = g(t, \mathbf{y}, \mathbf{u}) + \lambda^T f(t, \mathbf{y}, \mathbf{u})$ may be written as a Hamiltonian equation

$$\mathbf{y}' = \frac{\partial}{\partial \lambda} H$$
$$\lambda' = -\frac{\partial}{\partial \mathbf{y}} H$$
$$\mathbf{u} = \operatorname{argmin}_{\mathbf{u}} H(t, \mathbf{y}, \lambda, \mathbf{u})$$

Pontryagin Maximum Principle derived

Coupling by Lagrangian Multipliers

$$\Phi = \int_0^T [g(t,y,u) - \lambda^T(y' - f(t,y,u))] \, \mathrm{d}t$$

$$\delta\Phi = \int_0^T [g_u \delta u + \lambda^T f_u \delta u] \, \mathrm{d}t +$$

$$\int_0^T [g_y \delta y - \lambda^T \delta y' + \lambda^T f_y \delta y] \, \mathrm{d}t$$

$$\delta\Phi = \int_0^T [g_u \delta u + \lambda^T f_u \delta u] \, \mathrm{d}t +$$

$$\int_0^T [g_y + \lambda'^T + \lambda^T f_y] \delta y \, \mathrm{d}t$$

Some examples

Time optimal control

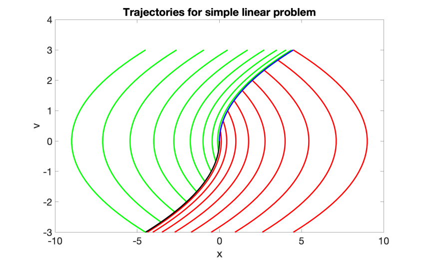
- The (OF) is just $\Phi = t$, i.e. $g = 1$. The end time T is a free parameter.
- We have to impose constraints on the controls.
- The optimal control is called bang-bang, it switches between min/max-values at certain points to ensure the boundary conditions.
- Example: Acceleration of mass point in 1D

$$\begin{aligned} x' &= v \\ v' &= u, \quad |u| \leq 1 \\ x(0) &= v(0) = 0, x(T) = x_1, v(T) = v_1 \text{ (BC)} \end{aligned}$$

Some examples

Time optimal control

- We start with control $u = 1$ (blue), and continue with $u = -1$ (red)
- So v is a (piecewise) linear function, x a quadratic function of t
- So x is quadratic in v, we can compute trajectories



- North to separatrix: use first $u = -1$ (black), then $u = +1$ (green)

Some examples

Energy optimal control

- Example: Acceleration of mass point in 1D
- The (OF) is given by $g = u^2/2$. We impose no constraints on the controls.
- The Hamiltonian is

$$H = u^2/2 + \lambda_1 v + \lambda_2 u$$

- We obtain $\lambda_1' = 0, \lambda_2' = \lambda_1, u = -\lambda_2$
- u is linear, v is quadratic, x is the cubic interpolating spline!
- The end time T must be fixed because $\lim_{T\to\infty} \Phi = 0$.

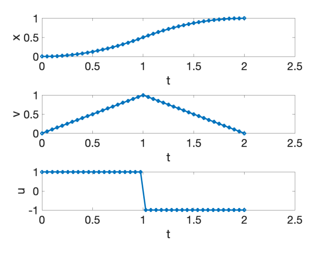
Hochschule für Technik und Wirtschaft Dresden
University of Applied Sciences

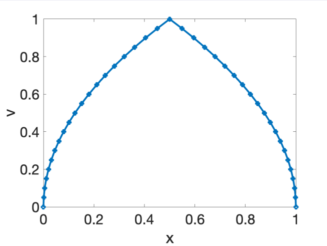
The linear problem with bang-bang-control

Accellerate and break, cubic spline predictor: matlab `fmincon`

$$x' = v$$
$$v' = u, |u| \leq 1$$

Try $(x_0, v_0) = (0, 0)$, $(x_1, v_1) = (1, 0)$, $T \to \min$.
After 30 Iterations we get $T = 2.000003$ with 40 intervals



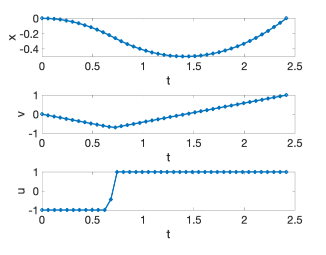


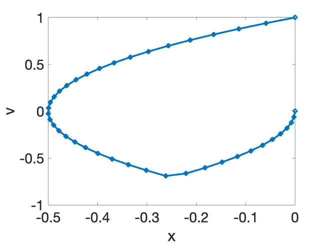
The linear problem with bang-bang-control

Accellerate, linear predictor: matlab `fmincon`

$$\begin{aligned} x' &= v \\ v' &= u, |u| \leq 1 \end{aligned}$$

Try $(x_0, v_0) = (0, 0)$, $(x_1, v_1) = (0, 1)$, $T \to \min$. After 24 Iterations we get $T = 2.4153$ with 40 intervals



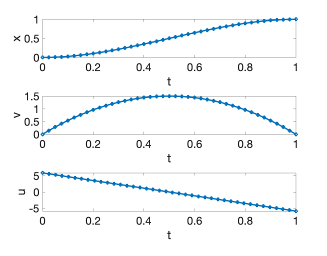


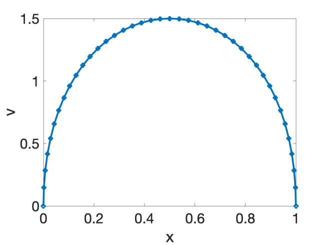
The linear problem with energy-control

Accellerate and break

$x' = v, v' = u, T \leq 1$ with $\int_0^T u^2 \, \mathrm{d}t \to \min$
Boundary values $(x_0, v_0) = (0, 0)$, $(x_1, v_1) = (1, 0)$.
Result is $T = 1$ (start with $T = 2$) with 40 intervals where

- Linear predictor: 17 Iterations.
- Cubic spline predictor: 9 Iterations.



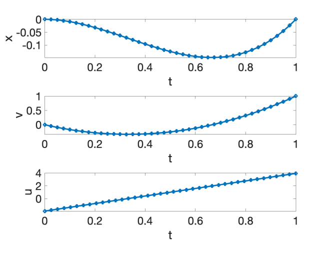


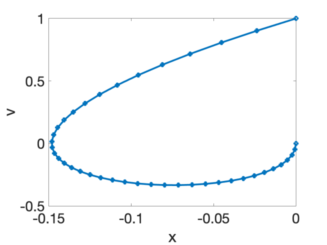
The linear problem with energy-control

Accellerate

$x' = v, v' = u, T \leq 1$ with $\int_0^T u^2 \, dt \to \min$
Boundary values $(x_0, v_0) = (0, 0)$, $(x_1, v_1) = (0, 1)$. Result is $T = 1$ (start with $T = 2$) with 40 intervals where

- Linear predictor: 17 Iterations.
- Cubic spline predictor: 7 Iterations.



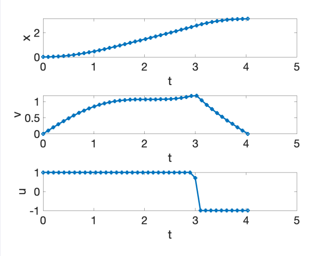


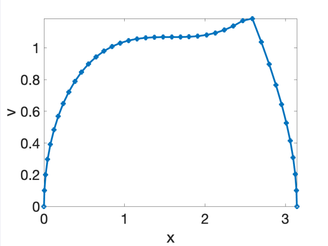
The pendulum with bang-bang control

Path: Move upwards, Torque: high.

$x' = v, v' = -\sin x + u, |u| \leq 1$ with $T \rightarrow \min$
Boundary values $(x_0, v_0) = (0, 0)$, $(x_1, v_1) = (\pi, 0)$. Result is $T = 4.031860$ with 40 intervals where

- Linear predictor: 48 Iterations.
- Cubic spline predictor: 7 Iterations ($T = 4.031850$).
- Probably not a global optimum



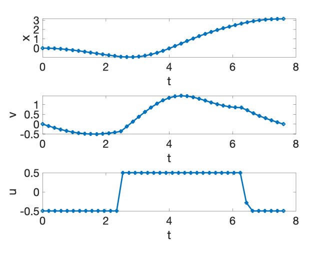


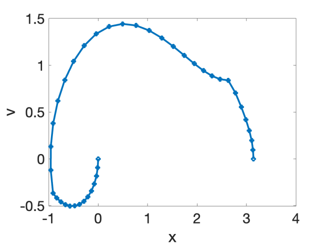
The pendulum with bang-bang control

Path: Move upwards, Torque: low.

$x' = v, v' = -\sin x + u, |u| \leq 0.5$ with $T \to \min$
Boundary values $(x_0, v_0) = (0, 0)$, $(x_1, v_1) = (\pi, 0)$. Result is $T = 7.6088$ with 40 intervals where

- Linear predictor: 925 Iterations.
- Cubic spline predictor: Solver failed.



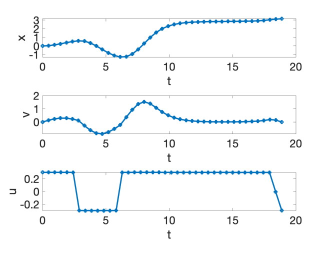


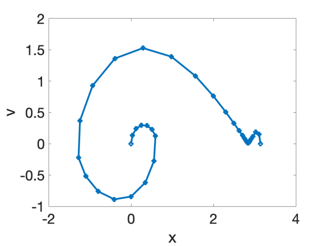
The pendulum with bang-bang control

Path: Move upwards, Torque: very low

$x' = v, v' = -\sin x + u, |u| \leq 0.3$ with $T \to \min$
Boundary values $(x_0, v_0) = (0, 0)$, $(x_1, v_1) = (\pi, 0)$ Result with 40 intervals

- Linear+Cubic predictor: Solver fails.
- Homotopy: start with $|u| < 0.5$, then decrease bound.
- Result: $T = 18.8$ by 925+842 Iterations.





Animation of biomechanical Systems

- Aim: Near-Reality Animation of motion of humans, anmials ...
- Model: Multibodysystem
  - Rigid bodies (Bones)
  - Constrainrs (Joints)
  - Forces (Muscels, Gravitation)

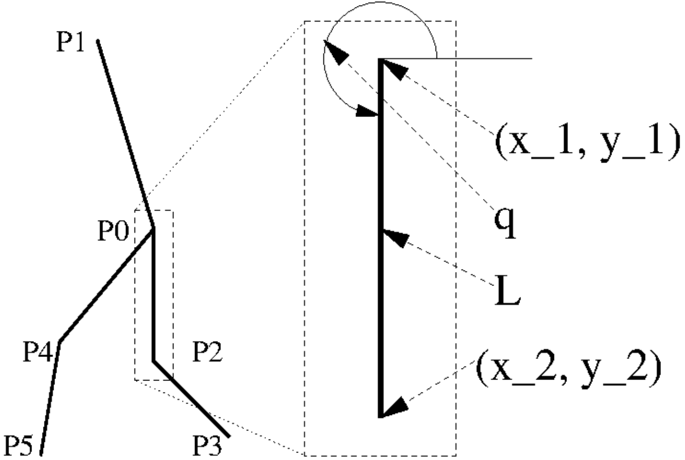
When muscle force known $\Rightarrow$

- Motion described by ODE
- Numerically solvable?

How compute muscle force= control?

HTWD Hochschule für Technik und Wirtschaft Dresden University of Applied Sciences

Geometrical Configuration



- Degrees of freedom: Angles q_1, q_2, q_3, q_4, q_5,
- Parameter: Length, masses, inertia
- Anchor point $P2$ (fix) $\Rightarrow$ $P0, P1, P3, P4, P5$ determined

Physical equations

- Independent degrees of freedom $q = (q_1, \ldots, q_N)$ (hier: Winkel q_i, $N = 5$)
- Determine kinetic (T) and potentia; Energy (V)
- muscle force $\Rightarrow$ moments in joints
  - Moment $=$ generalized outer force
  - virtual work δA_i for small displacements δq_i

Hamiltonian-Prinziple: Integral over action $L(q, q') = T - V$ ist extremal for predefined initial and end position
Euler-Lagrange-equations:

$$\frac{d}{dt}\frac{\partial L(q, q')}{\partial q_i'} = \frac{\partial L(q, q')}{\partial q_i} + \frac{\partial A(q)}{\partial q_i}$$

Homogeous Stab

Energy for mass m, Nodes $P_i = (x_i, y_i)$, $P_j = (x_j, y_j)$

- $V = mg(y_i + y_j)/2$
- $T = m/6(x_i'^2 + x_i' x_2' + x_j'^2 + y_i'^2 + y_i' y_j' + y_j'^2)$

tree strukture, determine P_i:

- example: $P_3 = P_2 + (P_3 - P_2) = P_2 - l_o(\cos q_3, \sin q_3)$
- analogously: $P_0 = P_3 + (P_0 - P_3)$, $P_1 = P_0 + (P_1 - P_0)$, ...
- result: $x_i = x_i(q)$, $x_i' = x_i'(q, q')$

Virtual work with muscle force u_k zwischen q_{k_1}, q_{k_2}

- $\delta A_k = -(\delta q_{k_2} - \delta q_{k_1}) u_k$

Euler Lagrange equations

$$T = \tfrac{1}{2} q'^T M(q) q',\ V = V(q),\ A = q^T D u,\ L = T - V.$$

- Left hand side

$$\frac{d}{dt}\frac{\partial L}{\partial q'} = \frac{d}{dt} M(q) q' = M(q) q'' + \sum_i q_i' \frac{\partial M(q)}{\partial q_i} q'$$

- Right hand side

$$\frac{\partial L}{\partial q} + \frac{\partial A}{\partial q} = \frac{1}{2}\frac{\partial q'^T M(q) q'}{\partial q} - \frac{\partial V(q)}{\partial q} + Du$$

$$M(q) q'' = -\sum_i q_i' \frac{\partial M(q)}{\partial q_i} q' + \frac{1}{2}\frac{\partial q'^T M(q) q'}{\partial q} - \frac{\partial V(q)}{\partial q} + Du$$

Boundary value problem

ODE: $q'' = f(t, q, q', u)$

- Let $x := (q, v) = (q, q')$
- ODE of first order $q' = v$, $v' = f(t, q, v)$

One (half) step $[0, T] \Rightarrow$ boundary conditions

- rump: $q_1(0) = q_1(T)$, $v_1(0) = v_1(T)$
- Thighs switch: $q_2(0) = q_4(T)$, $q_4(0) = q_2(T)$, $v_2(0) = v_4(T)$, $v_4(0) = v_2(T)$
- underlegs switch: $q_3(0) = q_5(T)$, ...

Boundary value problem:
$x'(t) = F(t, x(t), u(t)), \quad B(x(0), x(T)) = 0.$
is a constraint to minimize objective. Here: energy optimal.

Discretize first, then optimize

- Splines on grid $0 = t_0 < t_1 < \cdots < t_N = T$, $t_i = ih$
  - $x(t_i) = x_i$, $t \in [t_i, t_{i+1}] : x(t) = x_i + \frac{x_{i+1} - x_i}{h}(t - t_i)$
  - $u(t_i + h/2) = u_{i+1/2}$, $t \in [t_i, t_{i+1}] : u(t) = u_{i+1/2}$
- Diskretise ODE in $t_i + h/2$:
  $x' = f(t, x, u) \to \frac{x_{i+1} - x_i}{h} = f(t_i + h/2, \frac{x_i + x_{i+1}}{2}, u_{i+1/2})$
- Evaluate objective function by quadrature $T/N \sum_{i=1}^{N} u_{i-1/2}^2$
- Include boundary condition $B(x_0, x_N) = 0$

Result: Nonlinear least squares problem with constraints, solution by standard software

Evaluation

Optimize first, then discretize

- Difficult to find initial guess
- Shooting does not work
- Multiple shooting?

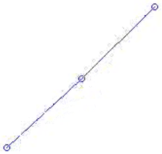
Discretize first, then optimize

- Difficult to find suitable initial guess
- Multiphase solution prozess neccessary: Multigrids
- Combine with homotopy $\Rightarrow$ Dynamic programming

Example: raise an arm

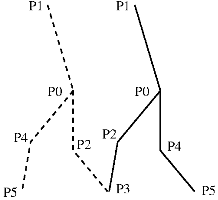
This is double pendulum with 2 controls:

$\Rightarrow$



Walk - initial setting

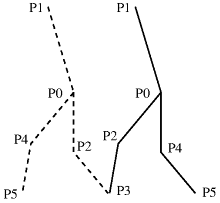
Configuration with rump, 2 tighs, 2 underlegs

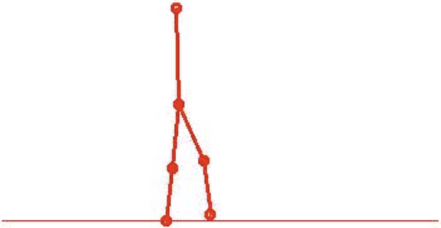


- We fix P3 and P5 at the bottom line at begin and end
- P2 and P4 change position relative to P0
- P3 and P5 change position relative to P0
- P1 is at same relative position in begin and end
- Thus: motion is periodic

Walk - 1. approach

- Configuration with rump, 2 tighs, 2 underlegs





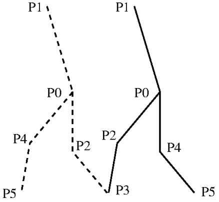
- Optimal solution: ...

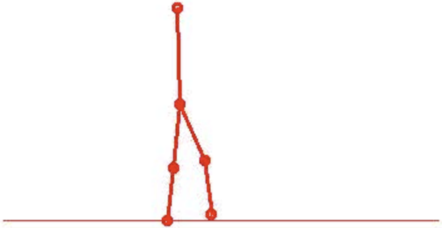
HTW Hochschule für Technik und Wirtschaft Dresden University of Applied Sciences

Walk - heads up!

Optimal solution: we hang down! Need further restrictions!

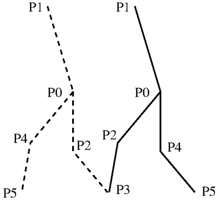
Head above 70% of size

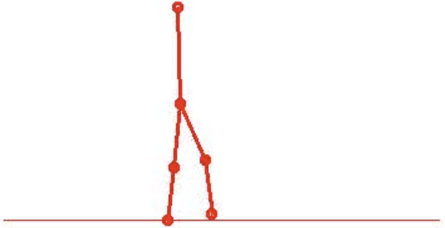




Walk - no stork

- restrict angle between thigh and underleg
- in each discretization point



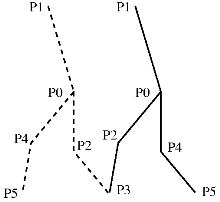


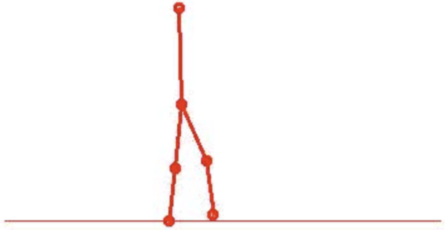
- reality near motion
- but: feet above ground $\Rightarrow$ further restriction

$\Rightarrow$ lower end time

Walk - results

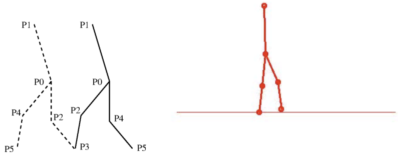
- Results for plane walk



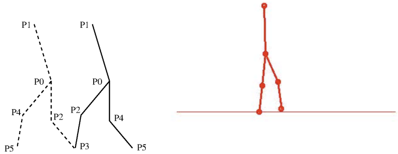


Walk - results

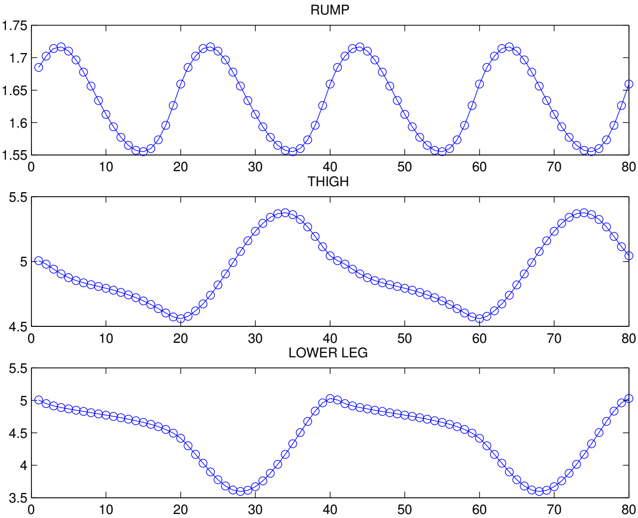
- Results for hilldown motion



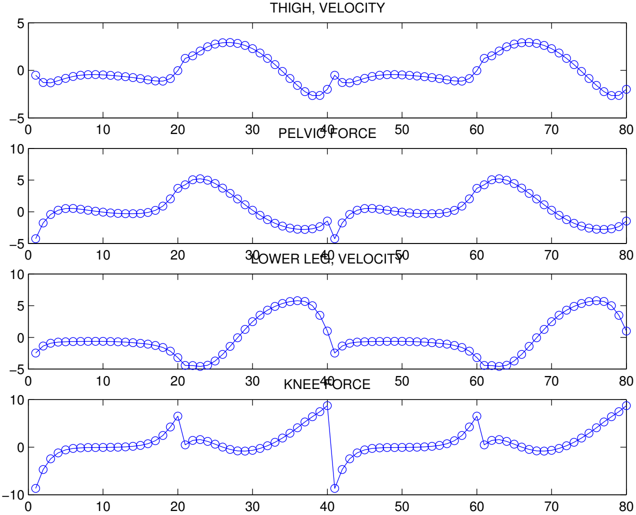
- results for more steep hillup motion



walk - motion data 1



walk - motion data 2

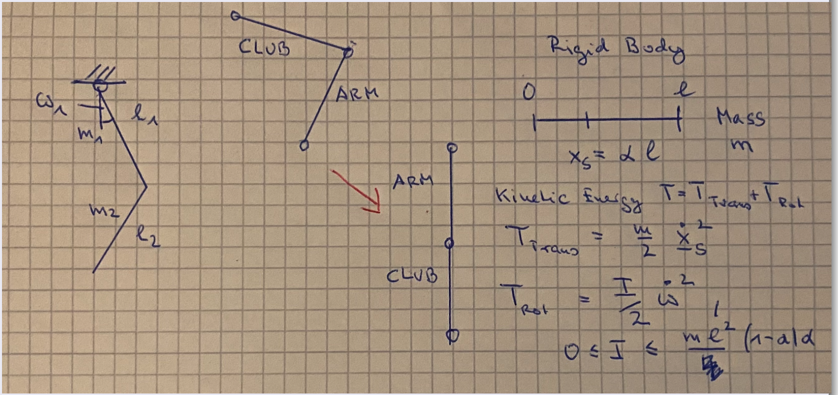


Outlook

- Application: Animation (fossils), Sports
- Automatic Generation of equations of motion
- Numerical solution: May be multiple shooting
- Dynamic programming to combine Multigrids and Homotopy
- Near reality muscle models
- One more application is coming...

Example: Golf swing

Geometrical configuration



Example: Golf swing

Measuring Callaway Big Bertha Iron 7

Head: 300g

Grip: 80g

Length:0.90m

Example: Golf swing

Big Bertha Iron 7 and my arm

	Iron 7	my Arm
mass m	0.38 kg	5kg
length l	0.9m	0.6m
α	0.8	0.44
β	0.08	0.1

where we have

$$\begin{aligned} x_s &= \alpha l \\ T &= \frac{m}{2}\dot{\mathbf{x}}_S^2 + \frac{m}{2} l^2 \dot{\phi}^2 \beta \end{aligned}$$

Example: Golf swing

Muscle force model

- Muscle force is essentially a torque u (moment of rotation)
- Easiest model: $|u_i| \leq U_i$
- More realistic The muscle force is decreasing with increasing velocity $v = \phi'$. For each muscle we have 3 parameters:
  - the maximum velocity $\bar{v}$ where the accelerating force becomes zero
  - the negative velocity $-\underline{v}$ where the decelerating force becomes maximal
  - the maximum decelerating force U_{max} which acts at that velocity $-\underline{v}$
  - we may have different values for protagonist and antagonist
  - interpolate a cubic spline between $-\underline{v}, \bar{v}$.

Example: Golf swing

Muscle force model

- From protagonist we have $u \leq u_{\max}$ with

$$u_{max}(v) = \begin{cases} U_{max} & \text{for } v \leq -\underline{v} \\ U_{max}\left(v + \frac{3\underline{v}+\bar{v}}{2}\right)\frac{(v-\bar{v})^2}{(\bar{v}+\underline{v})^3} & \text{for } -\underline{v} \leq v \leq \bar{v} \\ 0 & \bar{v} \leq v \end{cases}$$

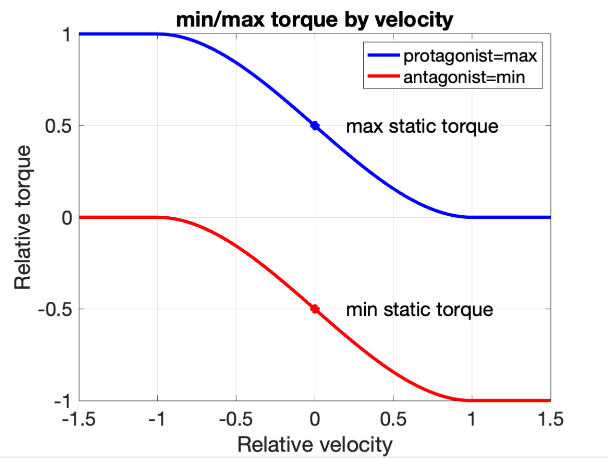
- From antagonist we have $u \geq u_{\min}$ with

$$u_{min}(v) = u_{max}(v) - U_{max}$$

where we assign the same parameters for simplicity.

Example: Golf swing

Muscle force model



Example: Golf swing

Sorry, work in progress

Thanks for the attention!