Scalable Parallel-in-Time with Multigrid: Theory, Applications, and Recent Developments

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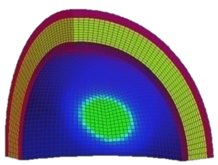
Outline

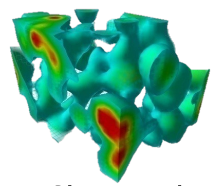
- Introduction to parallel-in-time with multigrid
  - Motivation
  - Multigrid reduction in time (MGRIT) overview
- Description of the XBraid parallel-in-time code
- Application areas
  - Focus: application of XBraid to deep learning
- Conclusions and future work

Multigrid is well suited for extreme scale

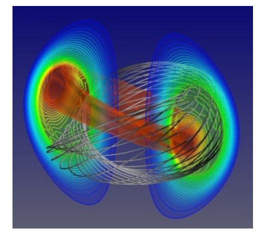
- For many applications, the fastest and most scalable solvers are already multigrid methods



Elasticity / Plasticity



Quantum Chromodynamics



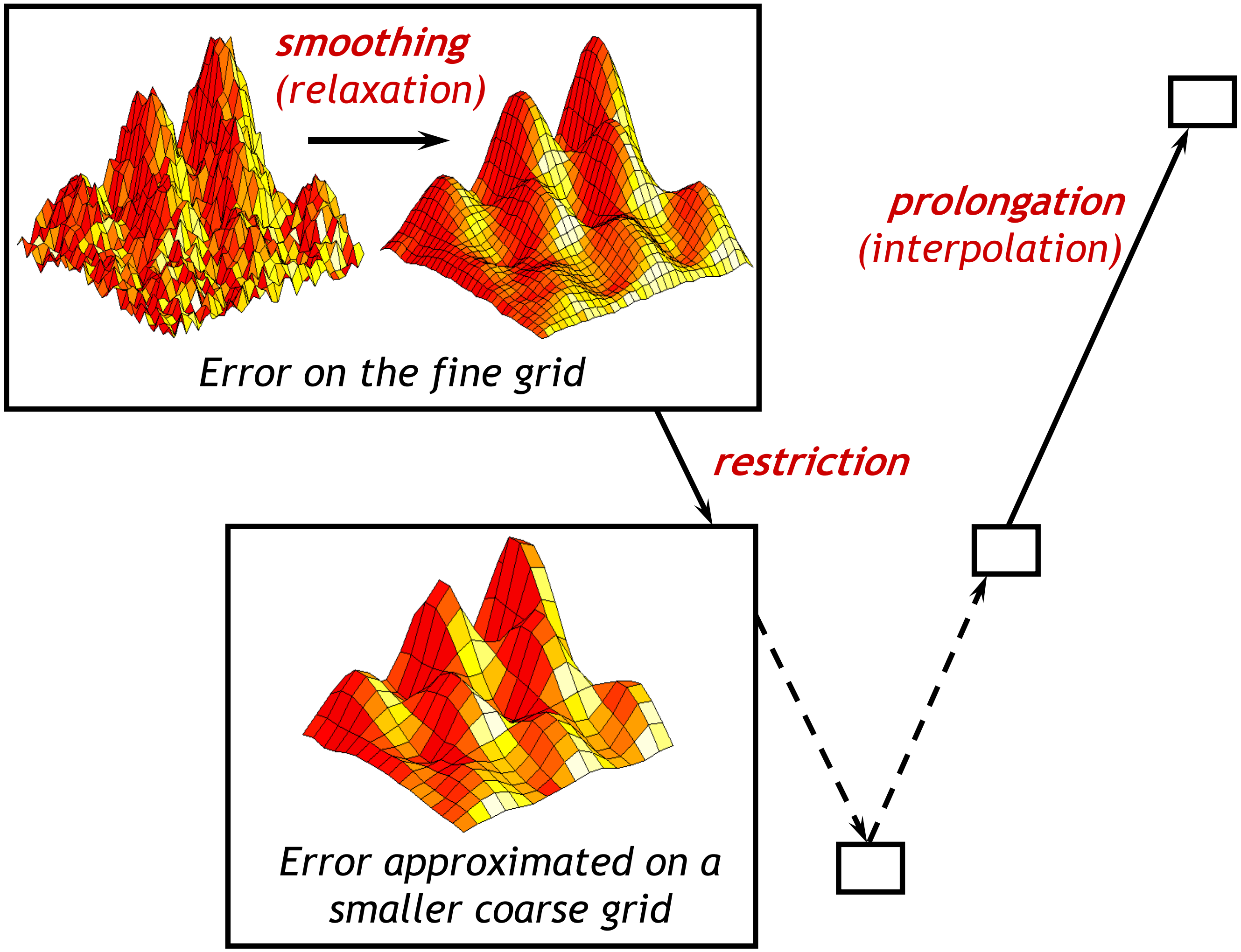
Magnetic Fusion Energy

- Extreme scale solvers need to:
  - Exhibit extreme levels of parallelism (e.g., billion or more cores)
    Spatial multigrid has already scaled to over 1 million cores
  - Minimize data movement
    Multigrid is $O(N)$ optimal
  - Exploit machine heterogeneity
    If the user's problem can exploit heterogeneity, then so can multigrid
  - Be resilient to faults
    Multigrid has already shown good resilience (iterative and multilevel helps)

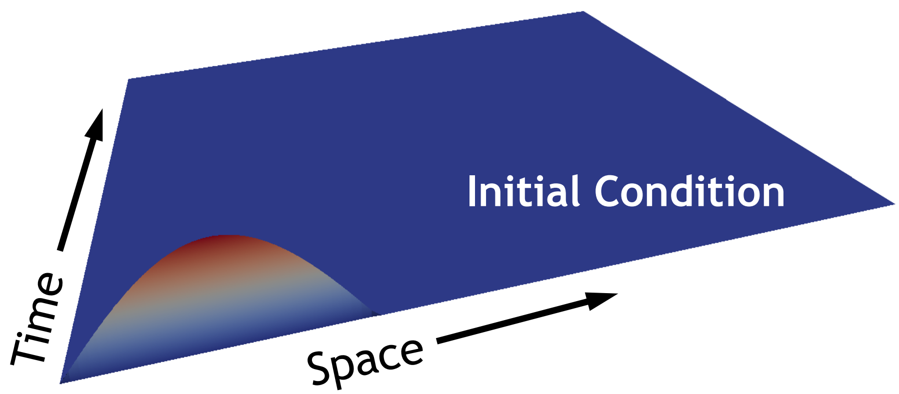
Parallel-in-time approach: Leverage spatial multigrid research

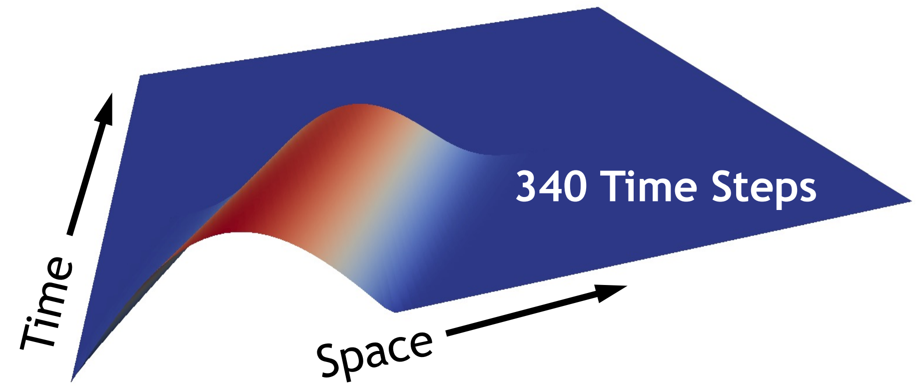
Solve:

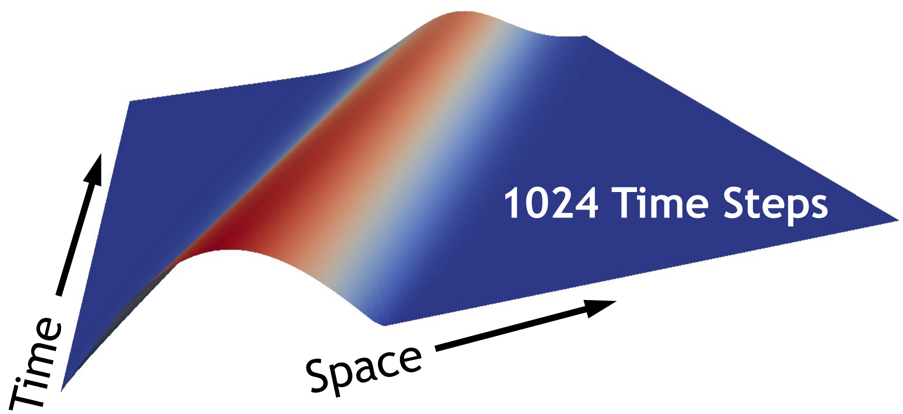
$A(u) = b$



Time stepping is sequential





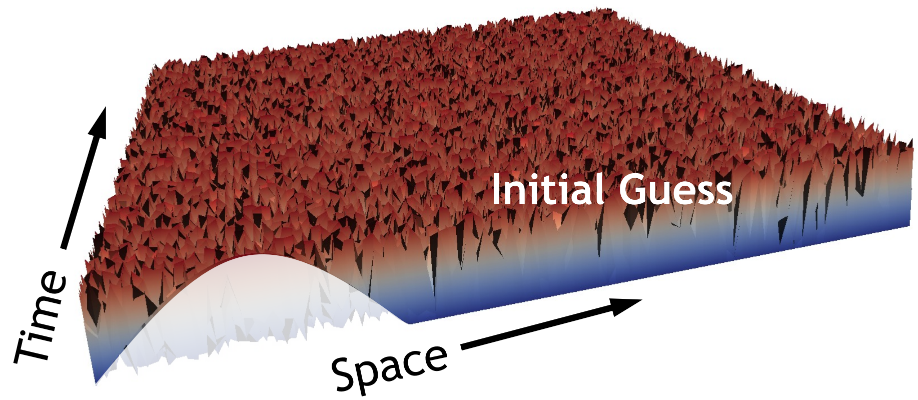


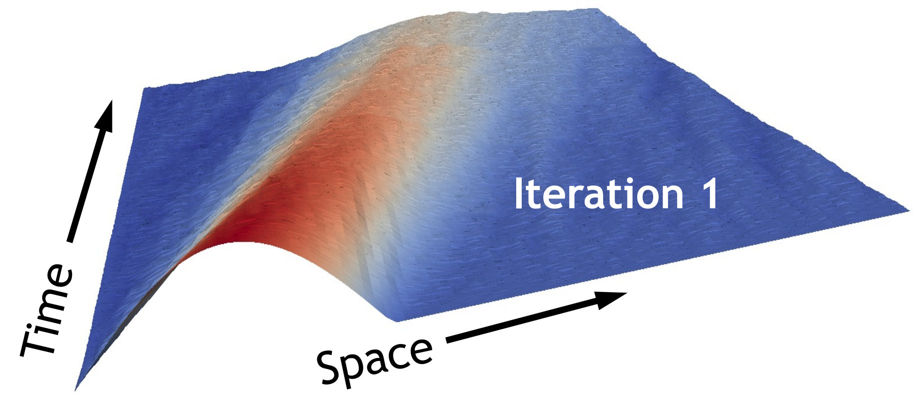
- Advection with 1st-order upwinding

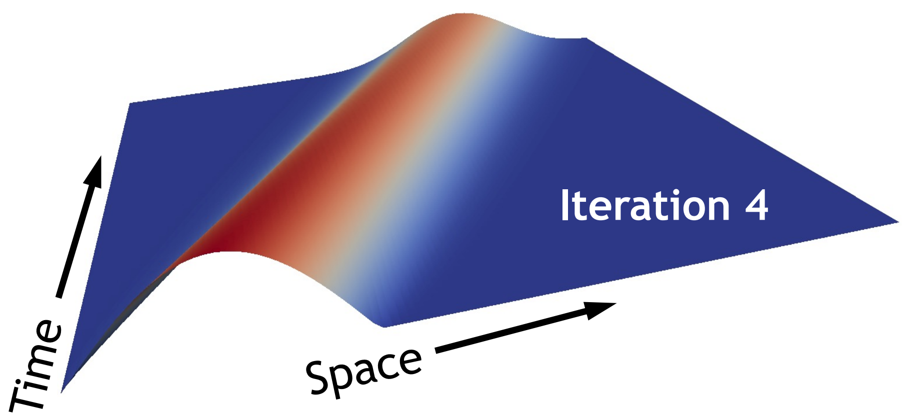
$$u_t = -cu_x$$

- Initial condition sine hump →propagates serially

Multigrid-in-time converges to serial solution in parallel







- Advection with 1st-order upwinding

$$u_t = -cu_x$$

- Initial condition sine hump
- Multilevel → fast data propagation
- Highly parallel, large speedups[1,2]
- Algorithmic research needed!

1. Gander, *50 Years of Time Parallel Time Integration*. Springer, 2015.
2. Ong, Schroder, *Applications of Time Parallelization*. CVS, Springer, 2020.

Technical approach

- Consider the **general** one-step method

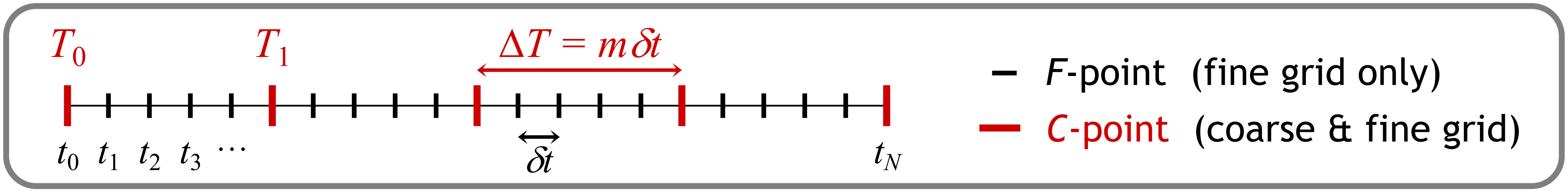
$$\boldsymbol{u}_i = \Phi_i(\boldsymbol{u}_{i-1}) + \boldsymbol{g}_i, \quad i = 1, 2, ..., N$$

- In the linear setting *(for simplicity)*, time marching $\equiv$ forward solve
  - This is an *O(N)* direct method, **but sequential**

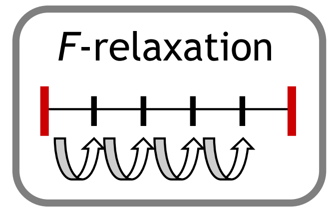
$$A\mathbf{u} \equiv \begin{pmatrix} I & & & \\ -\Phi & I & & \\ & \ddots & \ddots & \\ & & -\Phi & I \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_N \end{pmatrix} = \begin{pmatrix} \boldsymbol{g}_0 \\ \boldsymbol{g}_1 \\ \vdots \\ \boldsymbol{g}_N \end{pmatrix} \equiv \mathbf{g}$$

- Instead solve this system **iteratively** with a multigrid method
  - Extend multigrid reduction (MGR, 1979) to the time dimension
  - *O(N)*, highly parallel

Multigrid reduction in time (MGRIT)[1]



- Relaxation alternates between F- and C-points
  - F-relaxation is integration over coarse intervals (block Jacobi)



- Coarse system is a time rediscretization with fewer time-points
  - Main research question: approximate impractical Φ^m with Φ_Δ

$$A_\Delta = \begin{pmatrix} I & & & \\ -\Phi^m & I & & \\ & \ddots & \ddots & \\ & & -\Phi^m & I \end{pmatrix} \quad \Rightarrow \quad B_\Delta = \begin{pmatrix} I & & & \\ -\Phi_\Delta & I & & \\ & \ddots & \ddots & \\ & & -\Phi_\Delta & I \end{pmatrix}$$

  - Non-intrusive: time discretization unchanged, user only provides Φ

1. Falgout, Friedhoff, Kolev, MacLachlan, Schroder, *Parallel Time Integration with Multigrid*, SISC, 2014.

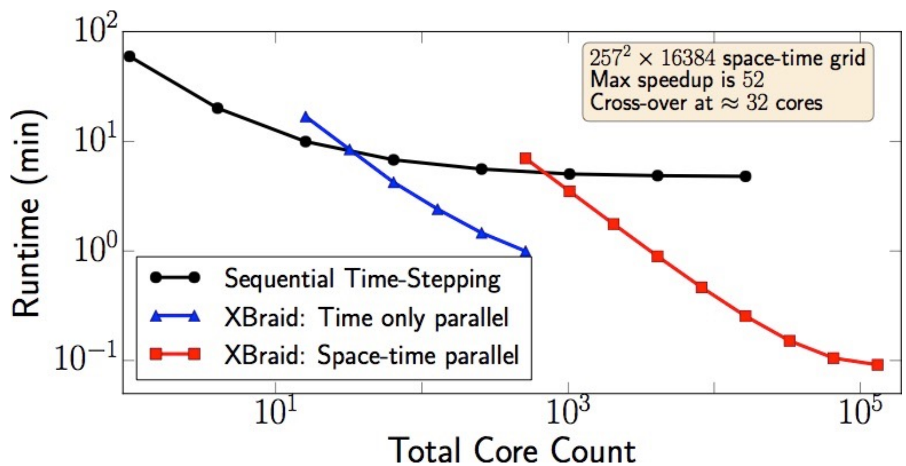
Broad summary of MGRIT

- Expose **concurrency** in the time dimension with multigrid
- **Non-intrusive**, with unchanged fine-grid problem
- Converges to **same solution** as sequential marching

$$\begin{pmatrix} I & & & \\ -\Phi & I & & \\ & \ddots & \ddots & \\ & & -\Phi & I \end{pmatrix}$$

- Optimal for variety of parabolic problems[1,2]
- Extends to **nonlinear** problems with FAS formulation
- In simple two-level setting, MGRIT $\equiv$ Parareal
- Sharp two-level[2] and multi-level[3] convergence theory

- Large speedups available, but in a new way
  - Time stepping is already O(N)
  - Useful only beyond a crossover
  - More time steps → more speedup potential



1. Falgout, Friedhoff, Kolev, MacLachlan, Schroder, *Parallel Time Integration with Multigrid*, SISC, 2014.
2. Dobrev, Kolev, Petersson, Schroder, *Two-level Convergence Theory for MGRIT*, SISC, 2017.
3. Hessenthaler, Southworth, Nordsletten, Rohrle, Falgout, Schroder, *Multilevel convergence analysis of MGRIT*, SISC, 2020.

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XBraid: Open source, non-intrusive, and flexible

- User also writes several wrapper routines:
  - *Step*, *Init*, *Clone*, *Sum*, *SpatialNorm*, *Access*, *BufPack*, *BufUnpack*
  - *Coarsen*, *Refine* (optional, for spatial coarsening)

- Example: *Step*(***u, status***)
  - Advance vector *u* from time *tstart* to *tstop*

- Code stores only *C*-points to minimize storage
  - Memory multiplier per processor:
    ~*O(log N)* with time coarsening, *O(1)* with space-time coarsening

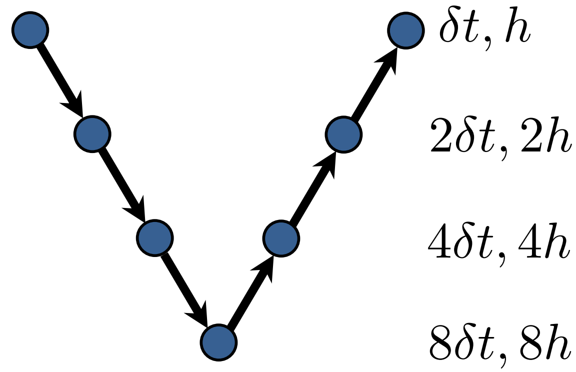
- Processes time-intervals to overlap communication and computation

- XBraid supports C, C++, Fortran, and Python interfaces

- XBraid has been integrated with various DOE codes
  (SunDials, GridDyn, Tycho2, and TorchBraid)
  https://github.com/XBraid/xbraid

XBraid Features

- Space-time coarsening speeds up computations and greatly reduces memory use[1]
  - User writes optional *Coarsen()* & *Refine()*
  - Works well for implicit parabolic



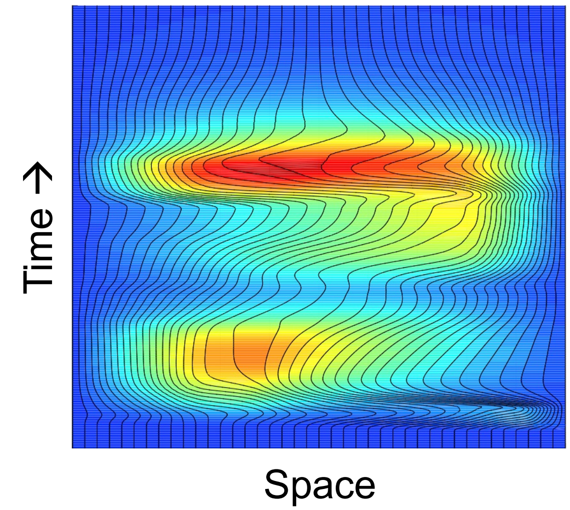
- Multi-step methods[2]
  - Convert to one-step method by grouping unknowns

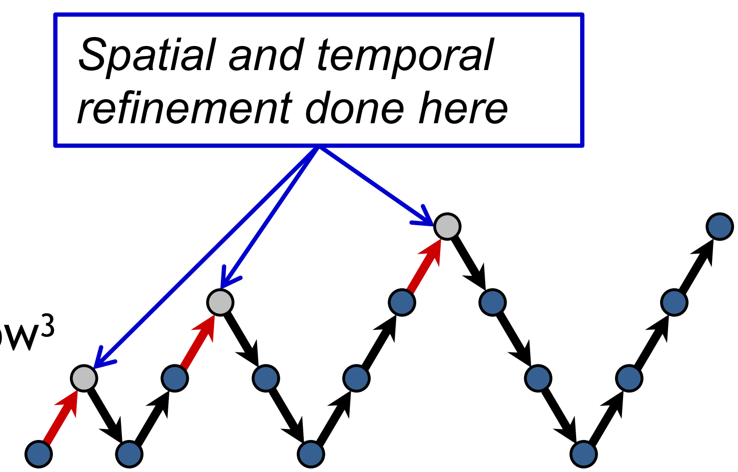
- Free Richardson-based extrapolation error estimator / correction[3]
  - Based on coarse time grid

1. Falgout, Manteuffel, O'Neill, S., *MGRIT for Nonlinear Parabolic Problems*, SISC, 2017.
2. Falgout, Lecouvez, Woodward, *A parallel-in-time algorithm for variable step multistep methods*, J. Comp. Sci., 2019.
3. Falgout, Manteuffel, O'Neill, Schroder, *Multigrid reduction in time with Richardson extrapolation*, ETNA, 2021.

XBraid Features: Adaptivity in time and space

- Moving spatial mesh[1]
  - 1D diffusion with time dependent source
  - Unsteady flow around moving cylinder
- Temporal refinement via Full Multigrid (FMG)
  - DAE power grid simulations in GridDyn[2] (52x speedup)
- Temporal and spatial refinement via FMG
  - 2D heat equation with FOSLS (6x speedup)
  - 13x speedup with space-time AMR for Couette Flow[3]
- Temporal load balancing under development





1. Falgout, Manteuffel, Schroder, Southworth, *Parallel-in-Time for Moving Meshes*, 2016. LLNL-TR-681918.
2. Schroder, Lecouvez, Falgout, Woodward, Top, *Parallel-in-Time Solution of Power Systems with Scheduled Events*, PES IEEE, 2018.
3. Christopher, Gao, Guzik, Falgout, Schroder, *Space-Time Parallel Alg. with Adaptive Mesh Refinement for CFD*, CVS Springer, 2020.

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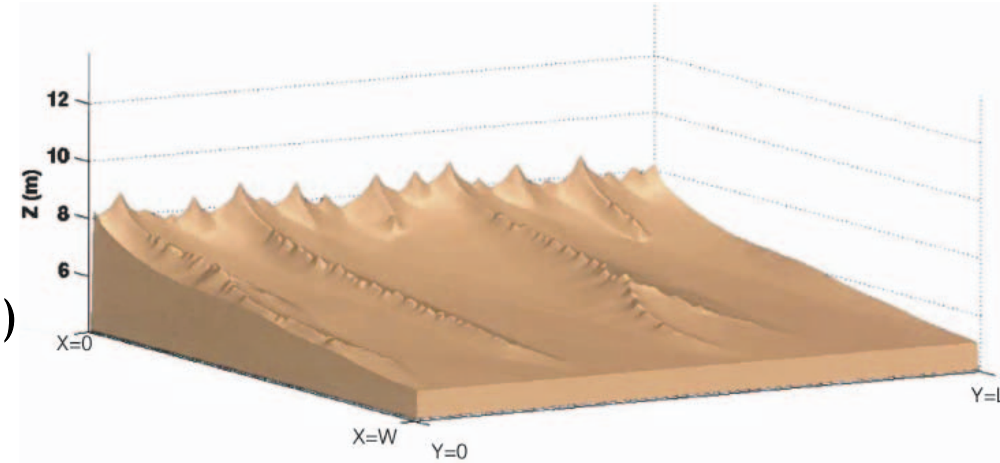
Experiments coupling our code XBraid with various application research codes

- Navier-Stokes (compressible and incompressible), Shallow Water
  - Strand2D, CarT3D, Cyclops, Chord
- Heat equation (including moving mesh example)
  - MFEM, hypre
- Elasticity (e.g., cardiac modeling)
  - CHeart
- Nonlinear diffusion, the *p*-Laplacian
  - MFEM
- Power-grid simulations
  - GridDyn+SunDials
- Explicit time-stepping coupled with space-time coarsening
  - Advection, Burger's Equation
  - MFEM
- Optimization (XBraid-adjoint), Machine Learning
  - CoDiPack, PyTorch (TorchBraid)

The p-Laplacian: nonlinear diffusion

- Solve $u_t = \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right)$
- 2D linear finite elements
  - 16K x 20K space-time problem
  - Backward Euler (Newton's method)
- Parallel results[1]
  - Crossover at ~40 processors in time
  - Speedup of 18x at 130K cores



Surface Erosion[2]

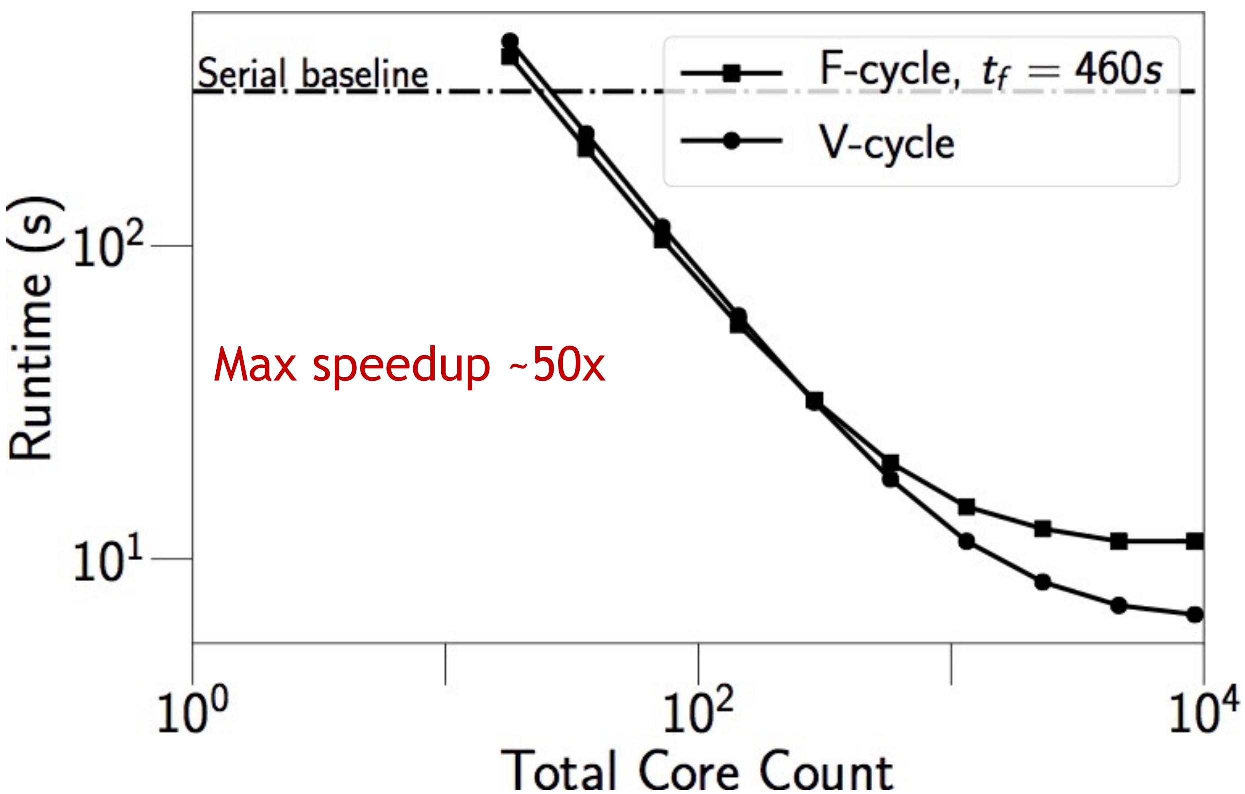
- Important parameters for performance
  - Full storage and space-time coarsening
  - Adjusting the Newton tolerance for the early iterations

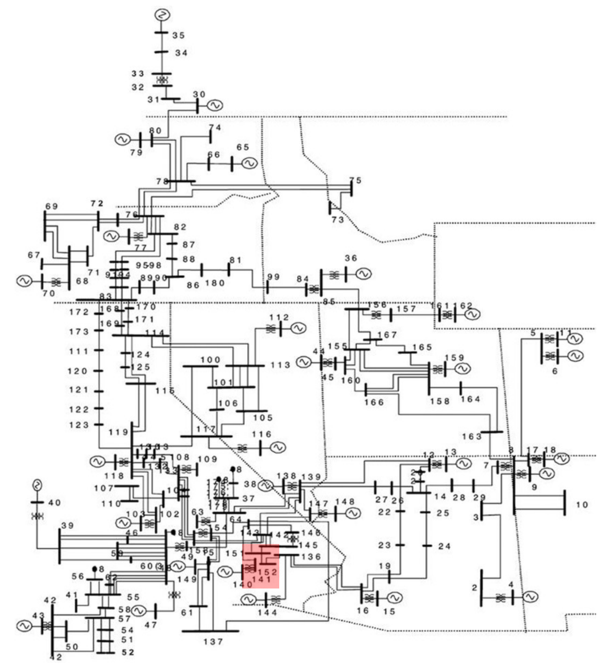
➔ In general, parallel-in-time works well for parabolic problems

1. Falgout, Manteuffel, O'Neill, S., *MGRIT for Nonlinear Parabolic Problems*, SISC, 2017.
2. Image courtesy of Birnir, Rowlett. *Mathematical Models for Erosion and the optimal Transportation of Sediment*. Int. J. Nonlinear Sci. Numer. Simul. 2013

Powergrid (DAE)

- Discontinuous square pulse applied to bus 141 every second[1]
  - Must handle discontinuities (events) for real-world relevance
  - Explore scalability w.r.t. number of discontinuities, 460s simulation has 460 events
  - Adaptively refine in time around discontinuities for improved accuracy





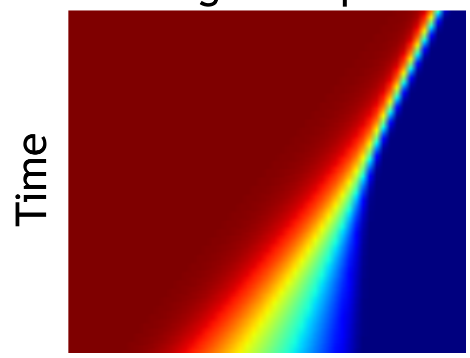
WECC System: 179 buses and 793 unknowns

1. Schroder, Lecouvez, Falgout, Woodward, Top, *Parallel-in-Time Solution of Power Systems with Scheduled Events*, PES IEEE, 2018.

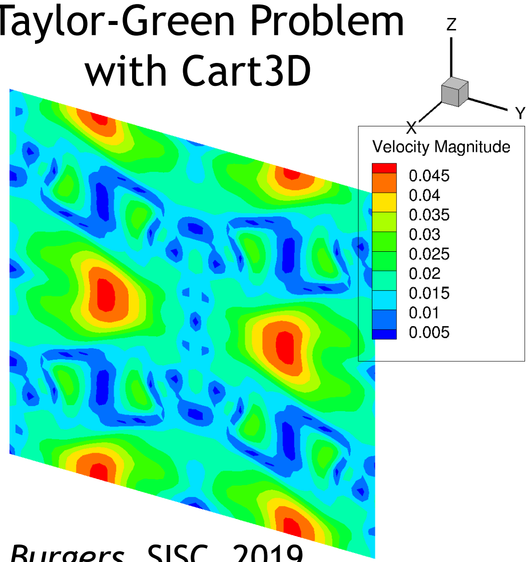
Hyperbolic problems and fluid dynamics

- Pure hyperbolic problems need significant artificial dissipation for **standard** MGRIT, parareal and other methods to work well
  - Expensive cycling (F-cycles) and expensive relaxation (FCF / FCFCF) also needed[1]
- For **modest** Reynolds numbers, success is more easily attained for CFD
- Navier-Stokes in 2D and 3D[2,3,4]
  - Multiple codes: Strand2D, Cart3D, CHeart, Chord
  - Compressible and incompressible

1D Burgers' Equation



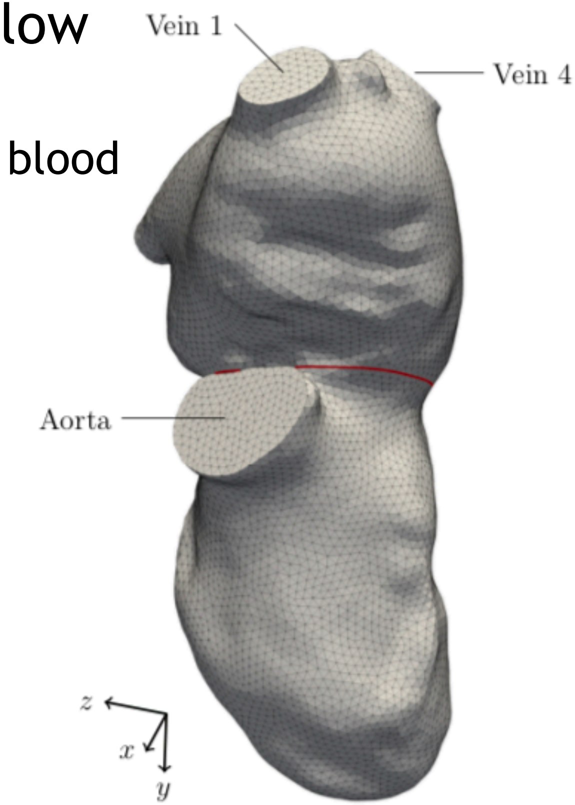
Taylor-Green Problem with Cart3D



1. De Sterck, Howse, Schroder, et al., *Parallel-in-Time MG with Adaptive Coarsening for Inviscid Burgers*, SISC, 2019.
2. Falgout, Katz, Kolev, Schroder, Wissink, Yang, *Parallel Time Integration with MGRIT for Compressible Fluid Dyn.*, 2014.
3. Christopher, Gao, Guzik, Falgout, Schroder, *Space-Time Parallel Alg. with Adaptive Mesh Refinement for CFD*, CVS Springer, 2020.
4. Christopher, Gao, Guzik, Falgout, Schroder, *Fully Parallelized Space-Time Adaptive Meshes for the Compressible Navier-Stokes Equations Using MGRIT, CVS Springer, 2020.*

Periodic fluid-structure interaction (FSI)

- Goal: speedup biomedical simulations, e.g., blood flow
  - Example problem: Periodic nonlinear flow in left ventricle
  - Equations: elasticity for solid deformations, Navier-Stokes for blood
- Periodicity allows for greater MGRIT efficiency[1]
  - MGRIT simulates only one periodic time interval
  - Standard method simulates many intervals until steady state
  - Modest Reynolds number of ~3000
- 20 processors in time → 5x speedup
- Leveraged multilevel convergence theory[2] to inform algorithm development

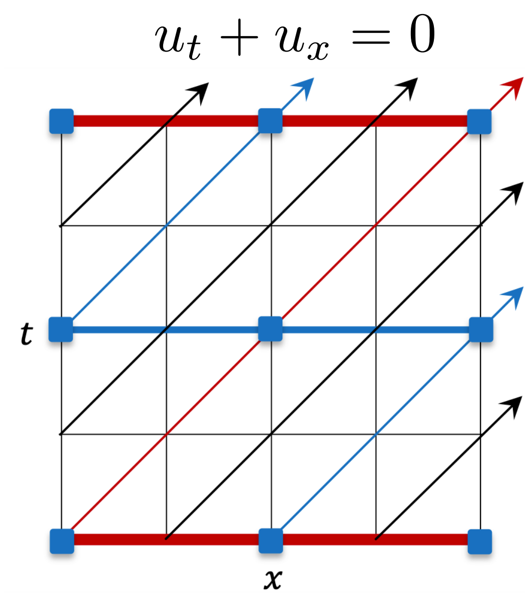


Left ventricle

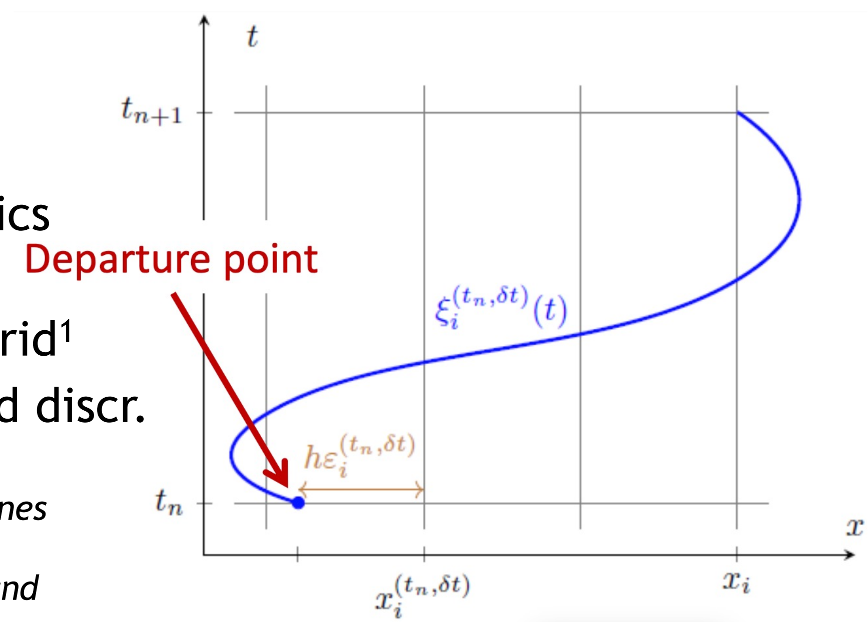
1. Hessenthaler, Falgout, Schroder, Nordsletten, Roehrle, *Time-Periodic Steady-State Solution of Fluid-Structure Interaction and Cardiac Flow Problems through MGRIT*. Comput. Meth. Appl. Mech. Eng., 2021.
2. Hessenthaler, Southworth, Nordsletten, Rohrle, Falgout, Schroder, *Multilevel convergence analysis of MGRIT*, SISC, 2020.

Pure hyperbolic: important recent progress

- Key insight: coarsen only in time, not space[2,3]
  → Do not coarsen away characteristics!
  - Coarse-grid time-stepping Φ_Δ should be semi-Lagrangian-like and follow characteristics



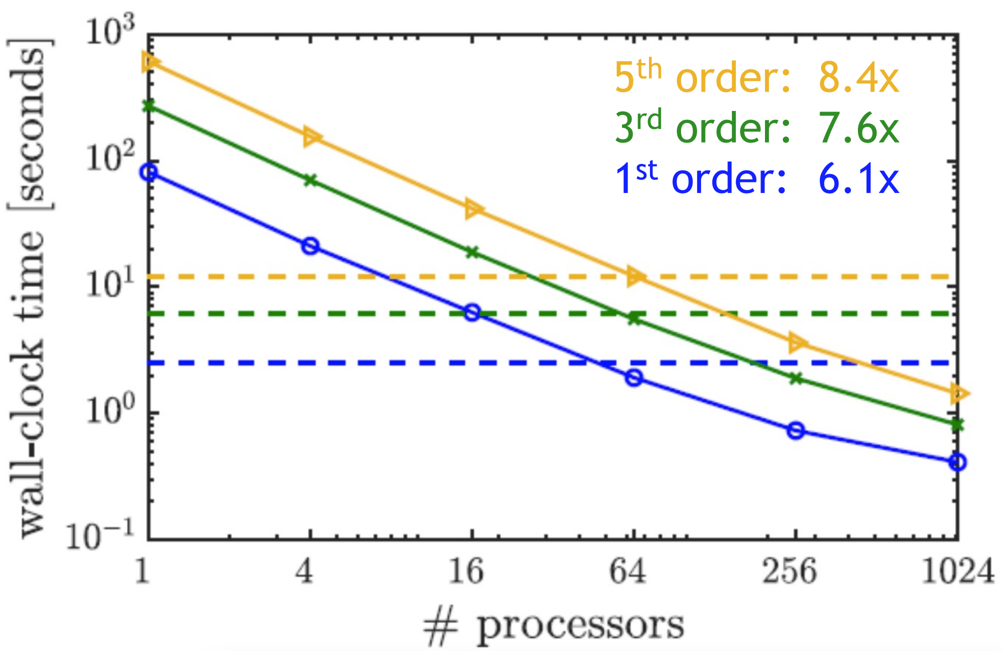
- Semi-Lagrangian Φ_Δ good for coarse-grid
  - Automatically adjust numerical domain of dependence in response to large coarse-grid time-step sizes
  - Integrate backwards along characteristics and interpolate
  - Paired with method-of-lines on finest-grid[1]
  - Precise matching of coarse and fine-grid discr.



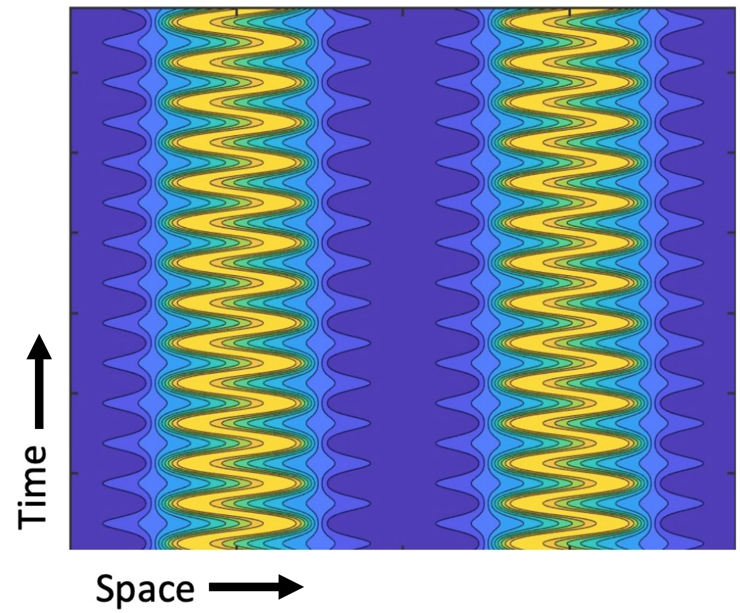
1. De Sterck, Falgout, Krzysik, Schroder, *Efficient MGRIT for Method-of-Lines Discretizations of Linear Advection*, J. Sci. Comput., 2023.
2. De Sterck, Falgout, Friedhoff, Krzysik, MacLachlan, *Optimizing MGRIT and Parareal coarse-grid operators for linear advection*, NLAA, 2021.
3. De Sterck, Falgout, Krzysik, *Fast MGRIT for Advection via Modified Semi-Lagrangian Coarse-Grid Operators*, SISC, 2023.

Pure hyperbolic: important recent progress

- Performance on linear advection excellent
  - Coarse-grid propagator designed to match the fine-grid propagator (not PDE)
  - If fine-grid time discretization is order p, then coarse-grid matches fine-grid discretization to order $p+1$
  - Effective for higher-order problems



$$u_t + \cos(2\pi t)\cos(2\pi x)u_x = 0$$



Recent work extends this to 1D hyperbolic systems

Parallel-in-Time for Chaotic Problems

- Chaotic problems are exceedingly sensitive to perturbations
  → How do we construct a sufficiently accurate coarse-grid?
  → Improve the coarse-grid equation (FAS) for nonlinear multigrid[1]

- Classic FAS is a constant correction τ about current coarse solution v

$$B_\Delta(v_{new}) = f + \tau$$

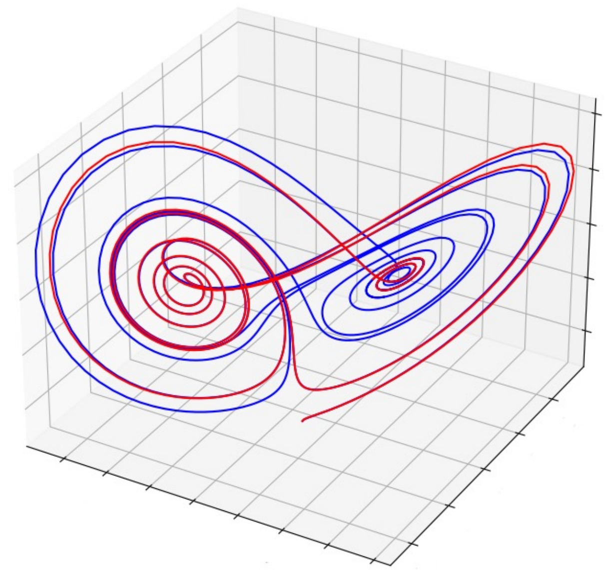
$$\tau_i = \Phi^m(v_i) - \Phi_\Delta(v_i)$$

- Delta-correction[2,3] $\mathbf{D}$ adds a linear correction

$$(B_\Delta + \mathbf{D})(v_{new}) = f + \tau + \mathbf{D}v$$

$$\mathbf{D}_i := (D_u\Phi^m - D_u\Phi_\Delta)(v_i)$$

→ Use low-rank approximation to $\mathbf{D}$ in practice



Lorenz System (butterfly)

1. Brandt, *Multi-level adaptive solutions to boundary-value problems*, Math. Comput., 1977
2. Yavneh, Dardyk, *A multilevel nonlinear method*, SISC, 2006.
3. Vargas, Falgout, Günther, Schroder, *Multigrid Reduction in Time for Chaotic Dynamical Systems*, SISC, 2024.

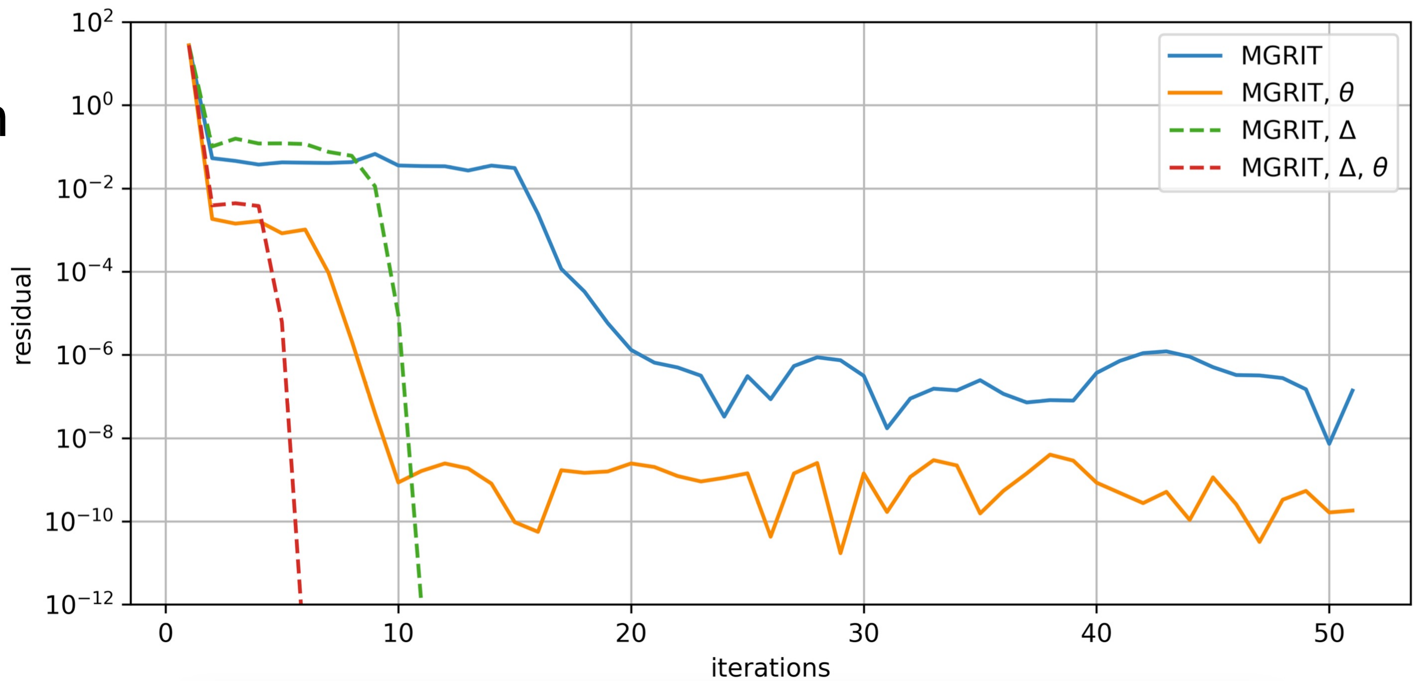
Results for Lorenz equation

- Further improve Φ_Δ by blending explicit and implicit methods
  - Blend so that coarse-grid matches fine-grid to higher-order
  - Coarse-grid propagator: Choose θ to match behavior of fine-grid

$$u' = f(u)$$

$$u_{i+1} = u_i + \theta\, hf(u_i) + (1-\theta)\, hf(u_{i+1})$$

- Results for Lorenz equation
  - 2-level
  - FCF-relaxation

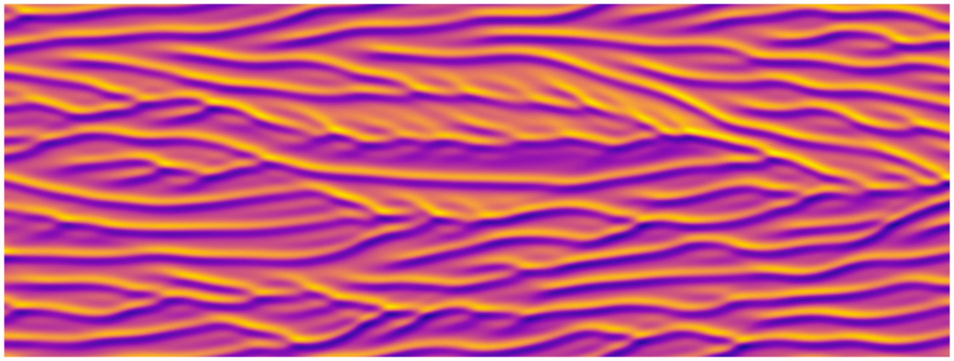


Results for Kuramoto-Sivashinsky (KS) equation

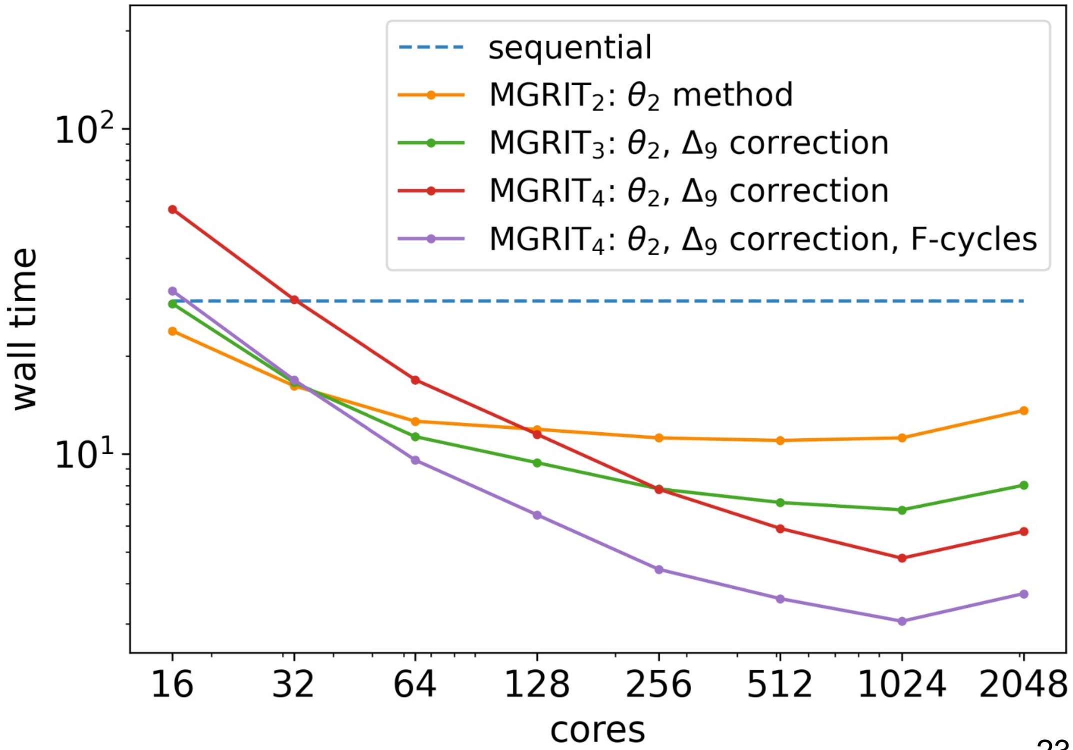
- KS equation

$$u_t = -u_{xx} - u_{xxxx} - u\, u_x$$

$$u(t,0) = u(t,L)$$



- Use low-rank Delta correction
  - Finitely many chaotic Lyapunov exponents
- Use 2nd order blended coarse-grid time-stepper (θ method)
- Speed-up ~10x
  - Over 10 Lyapunov time for true chaotic dynamics



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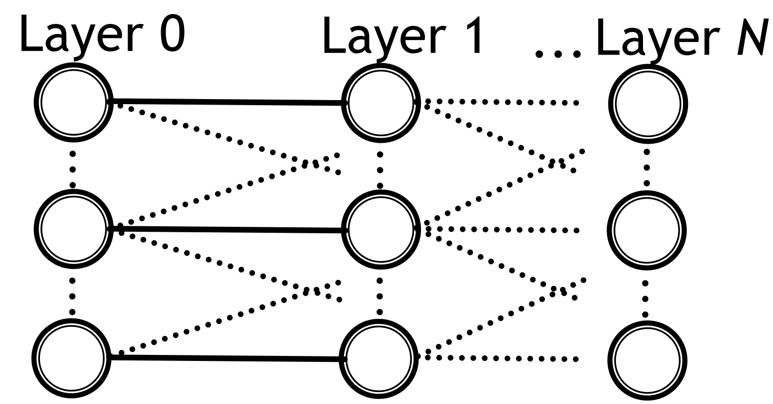
MGRIT for Deep Neural Networks (DNNs)

- DNNs are routinely used for many tasks
- However, training cost/times can be prohibitive (days, weeks, months...)
  - Due to the *many* forwards and backwards passes through the network

→ **Goal:** parallelize, speed up training



- Feed-forward network
  - Training pair: (y_{data}, c_{data})
  - W_n, b_n, y_n : Layer n weights, biases, state
  - **ResNet Propagation** (forward problem):

$$y_0 = y_{data}$$

$$y_{n+1} = y_n + F(W_n y_n + b_n) \quad \forall\, n = 0, \ldots, N-1$$

  - Learning problem:

$$\min_{W_n, b_n} \text{Loss}(y_N, c_{data}) \quad \text{subject to above forward problem}$$

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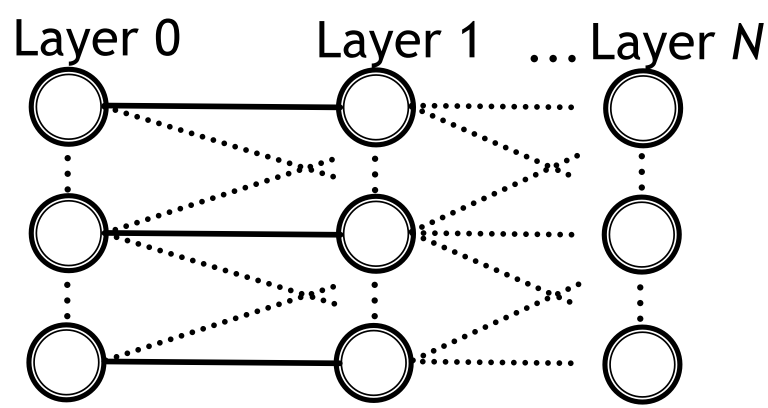

- Feed-forward network
  - Training pair: (y_{data}, c_{data})
  - W_n, b_n, y_n : Layer n weights, biases, state
  - **ResNet Propagation** (forward problem):

$$y_0 = y_{data}$$

$$y_{n+1} = y_n + F(W_n y_n + b_n) \quad \forall n = 0, \ldots, N-1$$

Insert time-step parameter !

MGRIT for DNNs

- Some popular deep networks (e.g., ResNets[1,2], Transformers[3,4], GRU[5]) have an equivalence to time-stepping
- Take ResNet architecture and augment with time-step parameter h

$$y_0 = y_{data}$$
$$y_{n+1} = y_n + hF(W_n y_n + b_n) \quad \forall\, n = 0, \ldots, N-1$$
$$\Leftrightarrow$$
$$y(0) = y_{data}$$
$$\frac{dy(t)}{dt} = F(W(t)y(t) + b(t)), \quad \forall t \in (0, T)$$

ResNet $\equiv$ Forward Euler discretization
Backprop $\equiv$ Discrete adjoint

- Training problem becomes

$$\min_{W(t),b(t)} \text{Loss}(y(T), c_{data}) \quad \text{subject to above ODE}$$

1. Haber, Ruthotto. *Stable Architectures for Deep Neural Networks*. Inverse Probl., 2017.
2. Weinan, *A Proposal on Machine Learning via Dynamical Systems*, Comm. Math. Stat., 2017.
3. *Understanding and improving transformer from a multiparticle dynamic system point of view*, Y. Lu et al., 2019. arXiv:1906.02762
4. *Stateful ODE-Nets using basis function expansions, A. Queiruga et al.*, Advances in Neural Information Processing Systems, 2021.
5. *Parallel Training of GRU Networks with a Multi-Grid Solver for Long Sequences*, E. Moon and E. C. Cyr, ICLR 2022.

MGRIT for DNNs

- Transformers are powerful (e.g., ChatGPT), where attention mechanisms capture long-range dependencies within sequences

- Transformers can also be extended to the layer-parallel setting
  - Ignoring pre- and post-processing
  - Transformers consist of encoder and decoder layers of the following form:

$$\mathbf{X}^{[n+1]} = \mathbf{X}^{[n]} + \mathbf{F}_{\text{enc}}(t_n, \mathbf{X}^{[n]}) \qquad \mathbf{X}^{[n+1]} = \mathbf{X}^{[n]} + \mathbf{F}_{\text{dec}}(t_n, \mathbf{X}^{[n]})$$

$$\mathbf{F}_{\text{enc}}(t_n, x) := \varphi_1(x) + \varphi_2\left(x + \varphi_1(x)\right) \qquad \mathbf{F}_{\text{dec}}(t_n, x) := \varphi_1(x) + \varphi_3\left(x + \varphi_1(x)\right) + \varphi_2\left(x + \varphi_1(x) + \varphi_3\left(x + \varphi_1(x)\right)\right)$$

  - Where $\mathbf{X}^{[n]}$ is the network state at layer n, and

$$\varphi_1 := \mathrm{SA} \circ \mathrm{LN}, \varphi_2 := \mathrm{MLP} \circ \mathrm{LN}, \varphi_3 := \mathrm{CA} \circ \mathrm{LN}$$

for self-attention (SA), cross-attention (CA), and layer-norm (LN)

Related ODE Transformer works:

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Insert time-step parameter[2] !   Insert time-step parameter[2] !

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    for self-attention (SA), cross-attention (CA), and layer-norm (LN)

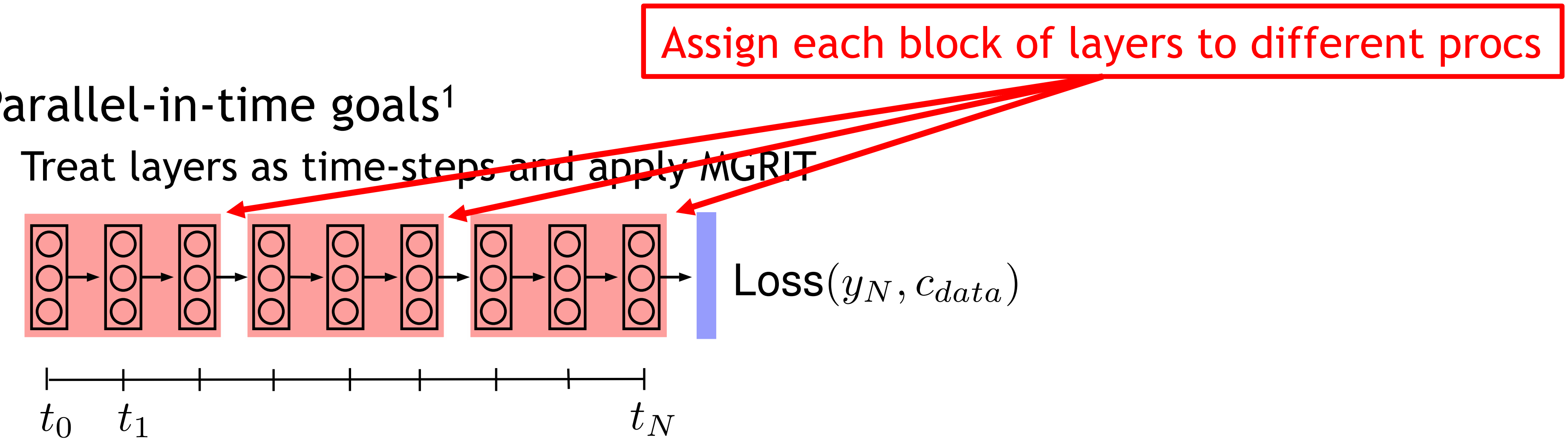
- Learning problem remains unchanged

Parallel-in-time and ODE-like neural networks

- Network propagation is equivalent to a forward Euler discretization, and backpropagation is equivalent to discrete adjoint!

  → *Remember:* Φ is layer-step in a DNN
  → *Use equivalence to apply XBraid to forward and backward problems*

- Parallel-in-time goals[1]
  - Treat layers as time-steps and apply MGRIT



  - Good strong and weak scaling with respect to number of network layers
    → Train a network with 5 layers with same wall-clock time as 1000 layers
  - Solve the same training problem (no shortcuts) as the sequential training version
  - Provide novel layer-parallelism (decoupled layer computations in parallel)

1. Günther, Ruthotto, Schroder, Cyr, Gauger. *Parallel-in-Layer Optimization for Training of Deep ResNets.* SIMODS, 2020.

Code: Layer-parallelism with multigrid

- **TorchBraid: XBraid (MGRIT code in MPI/C) and PyTorch**

  https://github.com/Multilevel-NN/torchbraid

  `$ pip install path/to/torchbraid`

- Python, C/MPI coupling with Cython (a little messy, but also impressive)

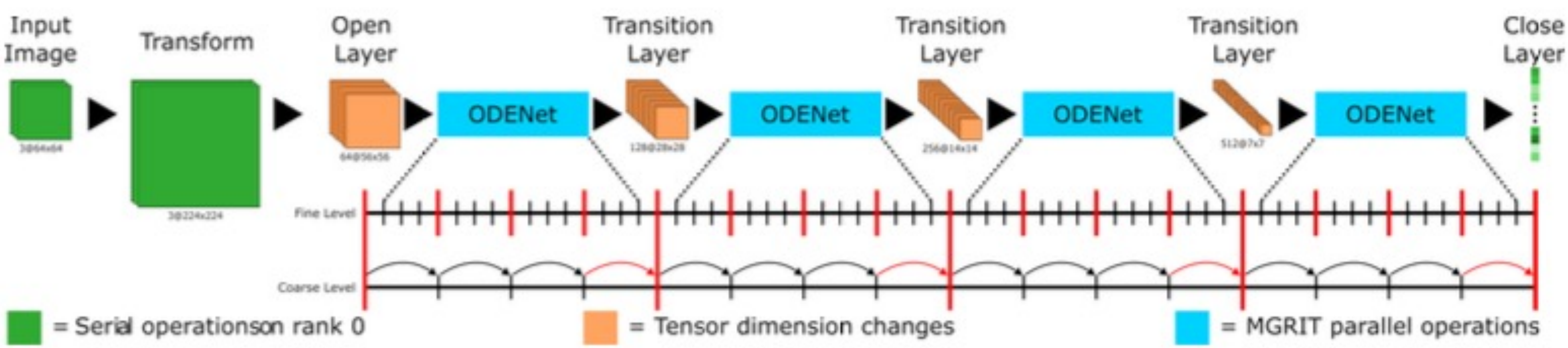
- Parallel Dataset and Dataloader functions inherited from PyTorch
  - Only load on root processor

- Compatible with data parallelism

- GPU-to-GPU direct communication key for performance
  - Requires CUDA-capable MPI, which you can test with
    `$ make tests -direct -gpu`

Results: Tiny ImageNet

- Extend TorchBraid ODENets to use max pooling and batch normalization
- Maintain max pooling layers as C-points on coarse levels
- Layer-parallel batch normalization
  - During inference, batch norm is identical to standard
  - During training, batch norm running averages are only updated during final fine-level evaluation at a layer

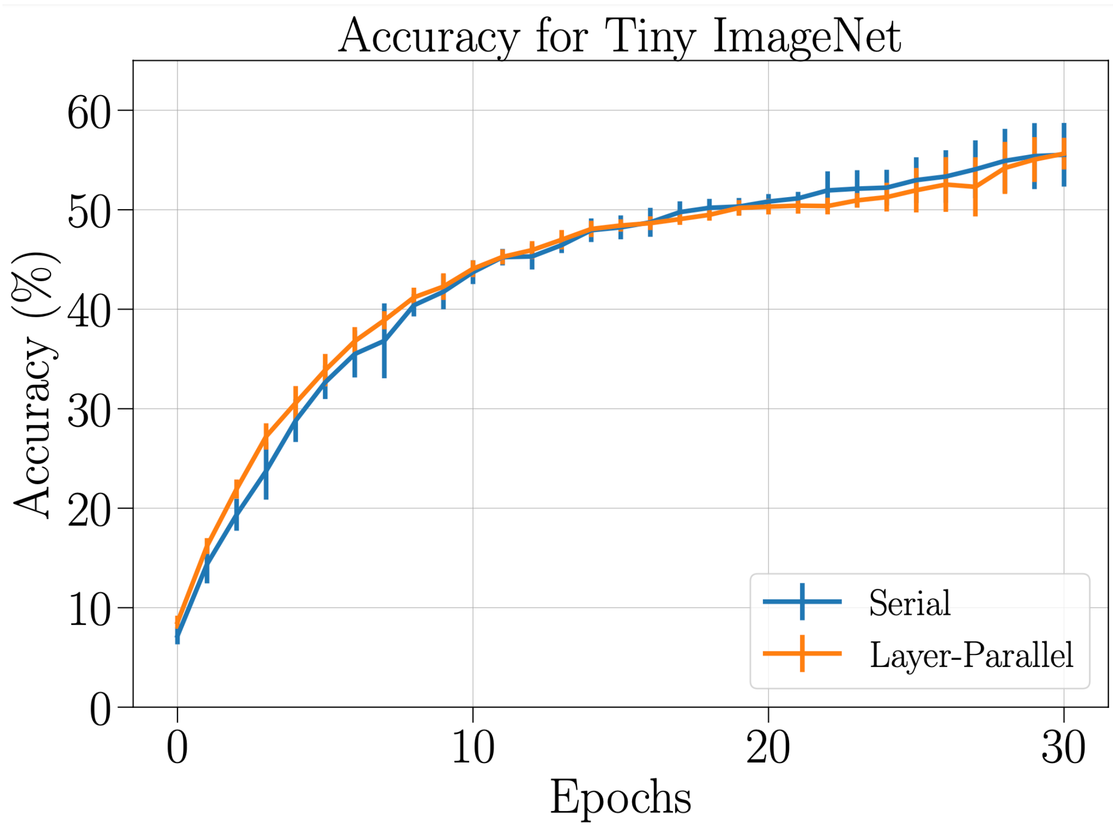


Results: Tiny ImageNet

- Train on PLEIADES cluster at University of Wuppertal
  - 5 compute nodes, each containing 8 A100 GPUs
  - Utilize GPUs for training and compare to serial time on GPU

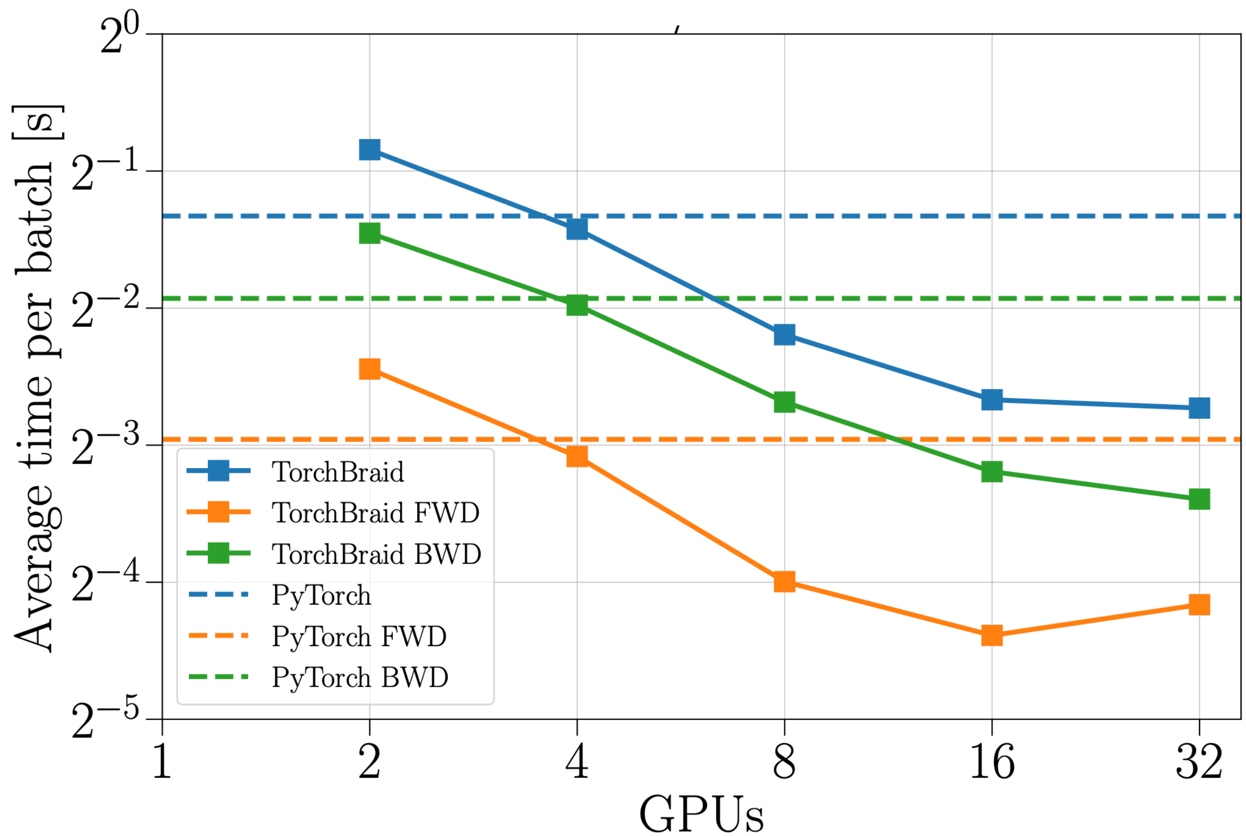
- Training setup
  - 256 total layers
  - 64 layers per ODENet block
  - 30 epochs and batch size of 50
  - 16 training runs
    - Vertical bars: std. dev

- Accuracy still good, but potential bias in layer-parallel gradient is current research topic



Results: Tiny ImageNet

- Strong scaling on Glinda
  - One A100 per node
  - Compare forward, backward, and overall times



- Crossover point: 4 GPUs

- 2.5x speedup at 16 GPUs

- Major coding effort! But, still work to be done on efficiency.
  - Transformer training is ongoing work

Nearly 50 years of research exists, but has only scratched the surface

- Earliest work goes back to 1964 by Nievergelt
  - Led to multiple shooting methods, Keller (1968)
- Space-time multigrid methods for parabolic problems
  - Hackbusch (1984); Horton (1992); Horton and Vandewalle (1995)
  - The latter is one of the first optimal & fully parallelizable methods to date
- Parareal was introduced by Lions, Maday, and Turincini in 2001
  - Probably the most widely studied method
  - Gander and Vandewalle (2007) show that parareal is two-level FAS multigrid
- Discretization specific work includes
  - Minion, Williams (2008, 2010) – PFASST, spectral deferred correction / parareal
  - De Sterck, Manteuffel, McCormick, Olson (2004, 2006) – FOSLS
- Research on these methods is ramping up!
  - Ruprecht, Speck, Schaufele, Götschel, Gander, De Sterck, …
    not an exhaustive list

Summary and conclusions

- Sequential time integration bottleneck is real
  - Parallel in time is needed for future architectures
  - This is a major paradigm shift
- We apply multigrid reduction to the time dimension
  - Multigrid is ideal for extreme scale (optimal, resilient, ...)
  - Result is a flexible and non-intrusive approach
  - Demonstrated speedups for a variety of problems
- There is much future work to be done!
  - More problem types, more complicated discretizations
  - Performance improvements, adaptive meshing
  - Enabling novel parallelism in machine learning
  - ...

Selected references

Parallel-in-Time

1. Falgout, Friedhoff, Kolev, MacLachlan, Schroder, *Parallel Time Integration with Multigrid*, SIAM J. Sci. Comput. (SISC), 2014.
2. Dobrev, Kolev, Petersson, Schroder, *Two-level Convergence Theory for MGRIT*, SIAM J. Sci. Comput. (SISC), 2017.
3. Guenther, Ruthotto, Schroder, Cyr, Gauger, *Layer-parallel training of deep residual neural networks*. SIAM J. Math. Data Sci. (SIMODS), 2020.
4. Sugiyama, Schroder, Southworth, Friedhoff, *Weighted Relaxation for Multigrid Reduction in Time*. Numer. Lin. Alg. Appl. Submitted, June 2021.
5. Ong, Schroder, *Applications of Time Parallelization*. CVS, Springer, 2020. *Review paper.*

Software

1. XBraid
   https://github.com/XBraid/xbraid
   Co-developed with Lawrence Livermore National Lab

2. TorchBraid
   https://github.com/Multilevel-NN/torchbraid
   Co-developed with Sandia National Lab