Error estimation for implicit-explicit general linear methods

Giuseppe Izzo [1]

1. Dipartimento di Matematica e Applicazioni "R.Caccioppoli",
Università degli Studi di Napoli Federico II, Italy

May 21, 2025

Part of this work is joint with Z.Jackiewicz (ASU, USA) and A. Cardone (Unisa, Italy)

Go20 Conference on Scientific Computing and Software

May 19-23, 2025, Marsalforn, Gozo, Malta

[1] e-mail: giuseppe.izzo@unina.it

Implicit-explicit General Linear Methods

Let us consider the following differential problem

$$\begin{cases} y'(t) = f\big(y(t)\big) + g\big(y(t)\big), & t \in [t_0, T], \\ y(t_0) = y_0 \in \mathbb{R}^d, \end{cases} \tag{1}$$

Where

- $f : \mathbb{R}^d \to \mathbb{R}^d$, represents the non-stiff processes
- $g : \mathbb{R}^d \to \mathbb{R}^d$, represents the stiff processes.

Implicit-explicit General Linear Methods

Let us consider the following differential problem

$$\begin{cases} y'(t) = f\big(y(t)\big) + g\big(y(t)\big), \quad t \in [t_0, T], \\ y(t_0) = y_0 \in \mathbb{R}^d, \end{cases} \tag{1}$$

Where

- $f : \mathbb{R}^d \to \mathbb{R}^d$, represents the non-stiff processes $\longleftarrow$ explicit method
- $g : \mathbb{R}^d \to \mathbb{R}^d$, represents the stiff processes. $\longleftarrow$ implicit method

The problem

$$\begin{cases} y'(t) = f\big(y(t)\big) + g\big(y(t)\big), \quad t \in [t_0, T], \\ y(t_0) = y_0 \in \mathbb{R}^d, \end{cases}$$

can be written as a partitioned system of the form

$$\begin{cases} x'(t) = \widehat{f}\big(x(t), z(t)\big), \\ z'(t) = \widehat{g}\big(x(t), z(t)\big), \qquad t \in [t_0, T], \\ x(t_0) = x_0, z(t_0) = z_0. \end{cases}$$

where

$$y = x + z, \quad y_0 = x_0 + z_0,$$

$$\widehat{f}(x,z) = f(x+z), \quad \widehat{g}(x,z) = g(x+z),$$

The problem

$$\begin{cases} y'(t) = f\big(y(t)\big) + g\big(y(t)\big), \quad t \in [t_0, T], \\ y(t_0) = y_0 \in \mathbb{R}^d, \end{cases}$$

can be written as a partitioned system of the form

$$\begin{cases} x'(t) = \widehat{f}\big(x(t), z(t)\big), \\ z'(t) = \widehat{g}\big(x(t), z(t)\big), \qquad t \in [t_0, T], \\ x(t_0) = x_0, z(t_0) = z_0. \end{cases}$$

where

$$\boxed{y = x + z}, \quad y_0 = x_0 + z_0,$$

$$\widehat{f}(x, z) = f(x + z), \quad \widehat{g}(x, z) = g(x + z),$$

Implicit-explicit General Linear Methods

$$\begin{cases} Y_i^{[n+1]} = h\sum_{j=1}^{i-1} a_{ij} f\left(Y_j^{[n+1]}\right) + h\sum_{j=1}^{s} \widehat{a}_{ij} g\left(Y_j^{[n+1]}\right) + \sum_{j=1}^{r} u_{ij} y_j^{[n]}, \; i = 1, 2, \ldots, s, \\ y_i^{[n+1]} = h\sum_{j=1}^{s} \left(b_{ij} f\left(Y_j^{[n+1]}\right) + \widehat{b}_{ij} g\left(Y_j^{[n+1]}\right)\right) + \sum_{j=1}^{r} v_{ij} y_j^{[n]}, \quad i = 1, 2, \ldots, r, \end{cases}$$

$n = 0, 1, \ldots, N-1$. Where

$$Y_i^{[n+1]} = y(t_n + c_i h) + O(h^{q+1}), \quad i = 1, 2, \ldots, s$$

Implicit-explicit General Linear Methods

$$\begin{cases} Y_i^{[n+1]} = h\sum_{j=1}^{i-1} a_{ij} f\left(Y_j^{[n+1]}\right) + h\sum_{j=1}^{s} \widehat{a}_{ij} g\left(Y_j^{[n+1]}\right) + \sum_{j=1}^{r} u_{ij} y_j^{[n]}, \ i = 1, 2, \ldots, s, \\ y_i^{[n+1]} = h\sum_{j=1}^{s} \left(b_{ij} f\left(Y_j^{[n+1]}\right) + \widehat{b}_{ij} g\left(Y_j^{[n+1]}\right)\right) + \sum_{j=1}^{r} v_{ij} y_j^{[n]}, \quad i = 1, 2, \ldots, r, \end{cases}$$

$n = 0, 1, \ldots, N-1$. Where

$$Y_i^{[n+1]} = y(t_n + c_i h) + O(h^{q+1}), \quad i = 1, 2, \ldots, s$$

and

$$y_i^{[n]} = \sum_{k=0}^{p} q_{ik} h^k x^{(k)}(t_n) + \sum_{k=0}^{p} \widehat{q}_{ik} h^k z^{(k)}(t_n) + O(h^{p+1}), \quad i = 1, 2, \ldots, r$$

Implicit-explicit General Linear Methods

$$\begin{cases} Y_i^{[n+1]} = h \sum_{j=1}^{i-1} \boxed{a_{ij}} f\left(Y_j^{[n+1]}\right) + h \sum_{j=1}^{s} \boxed{\widehat{a}_{ij}} g\left(Y_j^{[n+1]}\right) + \sum_{j=1}^{r} \boxed{u_{ij}} y_j^{[n]}, \; i = 1, 2, \ldots, s, \\ y_i^{[n+1]} = h \sum_{j=1}^{s} \left(\boxed{b_{ij}} f\left(Y_j^{[n+1]}\right) + \boxed{\widehat{b}_{ij}} g\left(Y_j^{[n+1]}\right) \right) + \sum_{j=1}^{r} \boxed{v_{ij}} y_j^{[n]}, \quad i = 1, 2, \ldots, r, \end{cases}$$

$n = 0, 1, \ldots, N-1$. Where

$$Y_i^{[n+1]} = y(t_n + \boxed{c_i} h) + O(h^{q+1}), \quad i = 1, 2, \ldots, s$$

and

$$y_i^{[n]} = \sum_{k=0}^{p} \boxed{q_{ik}} h^k x^{(k)}(t_n) + \sum_{k=0}^{p} \boxed{\widehat{q}_{ik}} h^k z^{(k)}(t_n) + O(h^{p+1}), \quad i = 1, 2, \ldots, r$$

Implicit-explicit General Linear Methods

$$\begin{cases} Y_i^{[n+1]} = h\sum_{j=1}^{i-1} a_{ij} f\left(Y_j^{[n+1]}\right) + h\sum_{j=1}^{s} \widehat{a}_{ij} g\left(Y_j^{[n+1]}\right) + \sum_{j=1}^{r} u_{ij} y_j^{[n]}, \; i = 1, 2, \ldots, \boxed{s}, \\ y_i^{[n+1]} = h\sum_{j=1}^{s} \left(b_{ij} f\left(Y_j^{[n+1]}\right) + \widehat{b}_{ij} g\left(Y_j^{[n+1]}\right)\right) + \sum_{j=1}^{r} v_{ij} y_j^{[n]}, \quad i = 1, 2, \ldots, \boxed{r}, \end{cases}$$

$n = 0, 1, \ldots, N-1$. Where

$$Y_i^{[n+1]} = y(t_n + c_i h) + O(\boxed{h^{q+1}}), \quad i = 1, 2, \ldots, \boxed{s}$$

and

$$y_i^{[n]} = \sum_{k=0}^{p} q_{ik} h^k x^{(k)}(t_n) + \sum_{k=0}^{p} \widehat{q}_{ik} h^k z^{(k)}(t_n) + O(\boxed{h^{p+1}}), \quad i = 1, 2, \ldots, \boxed{r}$$

Implicit-explicit General Linear Methods

$$\begin{cases} Y_i^{[n+1]} = h\sum_{j=1}^{i-1} a_{ij} f\left(Y_j^{[n+1]}\right) + h\sum_{j=1}^{s} \widehat{a}_{ij} g\left(Y_j^{[n+1]}\right) + \sum_{j=1}^{r} u_{ij} y_j^{[n]}, \; i = 1, 2, \ldots, s, \\ y_i^{[n+1]} = h\sum_{j=1}^{s} \left(b_{ij} f\left(Y_j^{[n+1]}\right) + \widehat{b}_{ij} g\left(Y_j^{[n+1]}\right)\right) + \sum_{j=1}^{r} v_{ij} y_j^{[n]}, \quad i = 1, 2, \ldots, r, \end{cases}$$

$n = 0, 1, \ldots, N-1$. Where

$$Y_i^{[n+1]} = y(t_n + c_i h) + O(h^{q+1}), \quad i = 1, 2, \ldots, s$$

and

$$y_i^{[n]} = \sum_{k=0}^{p} q_{ik} h^k x^{(k)}(t_n) + \sum_{k=0}^{p} \widehat{q}_{ik} h^k z^{(k)}(t_n) + O(h^{p+1}), \quad i = 1, 2, \ldots, r$$

- *Usually, starting and finishing procedures are needed,*

Implicit-explicit General Linear Methods

$$\begin{cases} Y_i^{[n+1]} = h\sum_{j=1}^{i-1} a_{ij} f\left(Y_j^{[n+1]}\right) + h\sum_{j=1}^{s} \widehat{a}_{ij} g\left(Y_j^{[n+1]}\right) + \sum_{j=1}^{r} u_{ij} y_j^{[n]}, \ i = 1, 2, \ldots, s, \\ y_i^{[n+1]} = h\sum_{j=1}^{s} \left(b_{ij} f\left(Y_j^{[n+1]}\right) + \widehat{b}_{ij} g\left(Y_j^{[n+1]}\right)\right) + \sum_{j=1}^{r} v_{ij} y_j^{[n]}, \quad i = 1, 2, \ldots, r, \end{cases}$$

$n = 0, 1, \ldots, N-1$. Where

$$Y_i^{[n+1]} = y(t_n + c_i h) + O(h^{q+1}), \quad i = 1, 2, \ldots, s$$

and

$$y_i^{[n]} = \sum_{k=0}^{p} q_{ik} h^k x^{(k)}(t_n) + \sum_{k=0}^{p} \widehat{q}_{ik} h^k z^{(k)}(t_n) + O(h^{p+1}), \quad i = 1, 2, \ldots, r$$

- *Usually, starting and finishing procedures are needed, but they can be avoided in particular cases.*

Introducing the vectors

$$y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix}, \quad Y^{[n+1]} = \begin{bmatrix} Y_1^{[n+1]} \\ \vdots \\ Y_s^{[n+1]} \end{bmatrix}, \quad f\big(Y^{[n+1]}\big) = \begin{bmatrix} f\big(Y_1^{[n+1]}\big) \\ \vdots \\ f\big(Y_s^{[n+1]}\big) \end{bmatrix},$$

$$g\big(Y^{[n+1]}\big) = \begin{bmatrix} g\big(Y_1^{[n+1]}\big) \\ \vdots \\ g\big(Y_s^{[n+1]}\big) \end{bmatrix}, \quad \mathbf{q}_k = \begin{bmatrix} q_{1k} \\ \vdots \\ q_{rk} \end{bmatrix}, \quad \widehat{\mathbf{q}}_k = \begin{bmatrix} \widehat{q}_{1k} \\ \vdots \\ \widehat{q}_{rk} \end{bmatrix},$$

and the matrices

$$\mathbf{A} = [a_{ij}], \;\; \widehat{\mathbf{A}} = [\widehat{a}_{ij}], \;\; \mathbf{U} = [u_{ij}], \; \mathbf{B} = [b_{ij}], \;\; \widehat{\mathbf{B}} = [\widehat{b}_{ij}], \;\; \mathbf{V} = [v_{ij}].$$

Implicit-explicit General Linear Methods

The method can be written in a more compact form

$$\begin{cases} Y^{[n+1]} = h(\mathbf{A} \otimes \mathbf{I})f\left(Y^{[n+1]}\right) + h(\widehat{\mathbf{A}} \otimes \mathbf{I})g\left(Y^{[n+1]}\right) + (\mathbf{U} \otimes \mathbf{I})y^{[n]}, \\ y^{[n+1]} = h(\mathbf{B} \otimes \mathbf{I})f\left(Y^{[n+1]}\right) + h(\widehat{\mathbf{B}} \otimes \mathbf{I})g\left(Y^{[n+1]}\right) + (\mathbf{V} \otimes \mathbf{I})y^{[n]}, \end{cases} \tag{2}$$

$n = 0, 1, \ldots, N-1$, $\mathbf{I} \in \mathbb{R}^d$.
Where

$$Y^{[n+1]} = y(t_n + \mathbf{c}h) + O(h^{q+1}),$$

and

$$y^{[n]} = \sum_{k=0}^{p} (\mathbf{q}_k \otimes \mathbf{I}) h^k x^{(k)}(t_n) + \sum_{k=0}^{p} (\widehat{\mathbf{q}}_k \otimes \mathbf{I}) h^k z^{(k)}(t_n) + O(h^{p+1}),$$

Implicit-explicit General Linear Methods

The method can be written in a more compact form

$$\begin{cases} Y^{[n+1]} = h(\mathbf{A}\otimes\mathbf{I})f\big(Y^{[n+1]}\big) + h(\widehat{\mathbf{A}}\otimes\mathbf{I})g\big(Y^{[n+1]}\big) + (\mathbf{U}\otimes\mathbf{I})y^{[n]}, \\ y^{[n+1]} = h(\mathbf{B}\otimes\mathbf{I})f\big(Y^{[n+1]}\big) + h(\widehat{\mathbf{B}}\otimes\mathbf{I})g\big(Y^{[n+1]}\big) + (\mathbf{V}\otimes\mathbf{I})y^{[n]}, \end{cases}$$

$n = 0, 1, \ldots, N-1$, $\mathbf{I} \in \mathbb{R}^d$.

Implicit-explicit General Linear Methods

The method can be written in a more compact form

$$\begin{cases} Y^{[n+1]} = h(\mathbf{A}\otimes\mathbf{I})f\left(Y^{[n+1]}\right) + h(\widehat{\mathbf{A}}\otimes\mathbf{I})g\left(Y^{[n+1]}\right) + (\mathbf{U}\otimes\mathbf{I})y^{[n]}, \\ y^{[n+1]} = h(\mathbf{B}\otimes\mathbf{I})f\left(Y^{[n+1]}\right) + h(\widehat{\mathbf{B}}\otimes\mathbf{I})g\left(Y^{[n+1]}\right) + (\mathbf{V}\otimes\mathbf{I})y^{[n]}, \end{cases}$$

$n = 0, 1, \ldots, N-1$, $\mathbf{I} \in \mathbb{R}^d$.
We are assuming that both methods, explicit and implicit, have the same coefficients matrices $\mathbf{U}$ and $\mathbf{V}$, and share the same abscissa vector $\mathbf{c}$.

Implicit-explicit General Linear Methods

The method can be written in a more compact form

$$\begin{cases} Y^{[n+1]} = h(\mathbf{A} \otimes \mathbf{I}) f\left(Y^{[n+1]}\right) + h(\widehat{\mathbf{A}} \otimes \mathbf{I}) g\left(Y^{[n+1]}\right) + (\mathbf{U} \otimes \mathbf{I}) y^{[n]}, \\ y^{[n+1]} = h(\mathbf{B} \otimes \mathbf{I}) f\left(Y^{[n+1]}\right) + h(\widehat{\mathbf{B}} \otimes \mathbf{I}) g\left(Y^{[n+1]}\right) + (\mathbf{V} \otimes \mathbf{I}) y^{[n]}, \end{cases}$$

$n = 0, 1, \ldots, N-1$, $\mathbf{I} \in \mathbb{R}^d$.
We are assuming that both methods, explicit and implicit, have the same coefficients matrices **U** and **V**, and share the same abscissa vector **c**.

IM, order p and stage order $q = p$

EX, order p and stage order $q = p$

Implicit-explicit General Linear Methods

The method can be written in a more compact form

$$\begin{cases} Y^{[n+1]} = h(\mathbf{A} \otimes \mathbf{I}) f\left(Y^{[n+1]}\right) + h(\widehat{\mathbf{A}} \otimes \mathbf{I}) g\left(Y^{[n+1]}\right) + (\mathbf{U} \otimes \mathbf{I}) y^{[n]}, \\ y^{[n+1]} = h(\mathbf{B} \otimes \mathbf{I}) f\left(Y^{[n+1]}\right) + h(\widehat{\mathbf{B}} \otimes \mathbf{I}) g\left(Y^{[n+1]}\right) + (\mathbf{V} \otimes \mathbf{I}) y^{[n]}, \end{cases}$$

$n = 0, 1, \ldots, N-1$, $\mathbf{I} \in \mathbb{R}^d$.
We are assuming that both methods, explicit and implicit, have the same coefficients matrices $\mathbf{U}$ and $\mathbf{V}$, and share the same abscissa vector $\mathbf{c}$.

$$\left.\begin{array}{l} \text{IM, order } p \text{ and stage order } q = p \\ \text{EX, order } p \text{ and stage order } q = p \end{array}\right. \Rightarrow \text{IMEX, order } p \text{ and stage order } q = p$$

Implicit-explicit General Linear Methods

$$y^{[n]} = \sum_{k=0}^{p} (\mathbf{q}_k \otimes \mathbf{I}) h^k x^{(k)}(t_n) + \sum_{k=0}^{p} (\widehat{\mathbf{q}}_k \otimes \mathbf{I}) h^k z^{(k)}(t_n) + O(h^{p+1})$$

Implicit-explicit General Linear Methods

$$y^{[n]} = \sum_{k=0}^{p} (\mathbf{q}_k \otimes \mathbf{I}) h^k x^{(k)}(t_n) + \sum_{k=0}^{p} (\widehat{\mathbf{q}}_k \otimes \mathbf{I}) h^k z^{(k)}(t_n) + O(h^{p+1})$$

can be written as

$$y^{[n]} = (\mathbf{W} \otimes \mathbf{I}) x(h, t_n) + (\widehat{\mathbf{W}} \otimes \mathbf{I}) z(h, t_n) + O(h^{p+1}),$$

where

$$x(h,t) = \begin{bmatrix} x(t) \\ hx'(t) \\ \vdots \\ h^p x^{(p)}(t) \end{bmatrix}, \quad z(h,t) = \begin{bmatrix} z(t) \\ hz'(t) \\ \vdots \\ h^p z^{(p)}(t) \end{bmatrix},$$

and

$$\mathbf{W} = \begin{bmatrix} \mathbf{q}_0 & \cdots & \mathbf{q}_p \end{bmatrix}, \quad \widehat{\mathbf{W}} = \begin{bmatrix} \widehat{\mathbf{q}}_0 & \cdots & \widehat{\mathbf{q}}_p \end{bmatrix}.$$

[1] G. Izzo and Z. Jackiewicz, Highly stable implicit-explicit Runge-Kutta methods, *Appl. Numer. Math.*, 113(2017) 71–92.

[2] M. Braś, G. Izzo, Z. Jackiewicz, Accurate Implicit-Explicit General Linear Methods with Inherent Runge-Kutta Stability, *J. Sci. Comput.* 70(2017),1105–1143.

[3] G. Izzo and Z. Jackiewicz, Transformed implicit-explicit DIMSIMs with strong stability preserving explicit part, *Numer. Alg.* 81(4), (2019), 1343–1359.

[4] G. Izzo and Z. Jackiewicz, Strong stability preserving implicit-explicit transformed general linear methods, *Math. Comput. Simul.*, (2019)

[5] G. Izzo and Z. Jackiewicz, Strong Stability Preserving IMEX Methods for Partitioned Systems of Differential Equations,
Spinger CAMC Vol.4(3), (2021), 719–754.

[1] G. Izzo and Z. Jackiewicz, Highly stable implicit-explicit Runge-Kutta methods, *Appl. Numer. Math.*, 113(2017) 71–92.

[2] M. Braś, G. Izzo, Z. Jackiewicz, Accurate Implicit-Explicit General Linear Methods with Inherent Runge-Kutta Stability, *J. Sci. Comput.* 70(2017),1105–1143.

[3] G. Izzo and Z. Jackiewicz, Transformed implicit-explicit DIMSIMs with strong stability preserving explicit part, *Numer. Alg.* 81(4), (2019), 1343–1359.

[4] G. Izzo and Z. Jackiewicz, Strong stability preserving implicit-explicit transformed general linear methods, *Math. Comput. Simul.*, (2019)

[5] G. Izzo and Z. Jackiewicz, Strong Stability Preserving IMEX Methods for Partitioned Systems of Differential Equations,
Spinger CAMC Vol.4(3), (2021), 719–754.

[6] G. Califano, G. Izzo, and Z. Jackiewicz, Starting procedures for general linear methods, *Appl. Numer. Math.* 120(2017), 165–175.

[7] G. Califano, G. Izzo, and Z. Jackiewicz, Strong stability preserving general linear methods with Runge-Kutta stability, *J. Sci. Comput* 76(2), (2018), 943–968.

[8] G. Izzo and Z. Jackiewicz, Strong stability preserving general linear methods, *J. Sci. Comput.* 65(2015), 271–298.

[9] G. Izzo and Z. Jackiewicz, Strong stability preserving transformed DIMSIMs, *J. Comput. Appl. Math.* 343 (2018), 174–188.

Example: Numerical Results for Shallow water model

$$\begin{cases} \dfrac{\partial}{\partial t}h + \dfrac{\partial}{\partial x}(hv) = 0, \\ \dfrac{\partial}{\partial t}(hv) + \dfrac{\partial}{\partial x}\left(h + \dfrac{1}{2}h^2\right) = \dfrac{1}{\varepsilon}\left(\dfrac{h^2}{2} - hv\right), \end{cases}$$

where h is the water height with respect to the bottom and hv is the flux.

We use periodic boundary conditions and initial conditions at $t_0 = 0$

$$h(0,x) = 1 + \frac{1}{5}sin(8\pi x), \quad hv(0,x) = \frac{1}{2}h(0,x)^2, \quad \text{with } x \in [0,1].$$

Example: Numerical Results for Shallow water model

$$\begin{cases} \dfrac{\partial}{\partial t}h + \dfrac{\partial}{\partial x}(hv) = 0, \\ \dfrac{\partial}{\partial t}(hv) + \dfrac{\partial}{\partial x}\left(h + \dfrac{1}{2}h^2\right) = \dfrac{1}{\varepsilon}\left(\dfrac{h^2}{2} - hv\right), \end{cases}$$

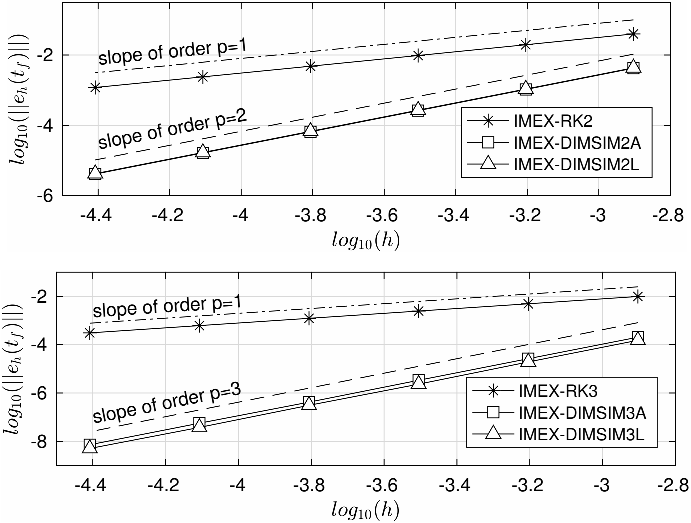
where h is the water height with respect to the bottom and hv is the flux.

We use periodic boundary conditions and initial conditions at $t_0 = 0$

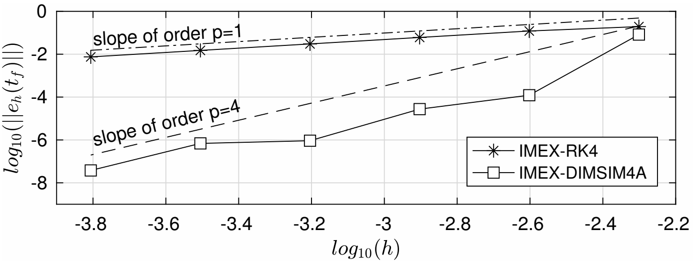
$$h(0,x) = 1 + \frac{1}{5}sin(8\pi x), \quad hv(0,x) = \frac{1}{2}h(0,x)^2, \quad \text{with } x \in [0,1].$$

The space derivative was discretized by a fifth order finite difference weighted essentially non-oscillatory (WENO5)

Shallow water model - NX = 201 - $\varepsilon = 10^{-6}$

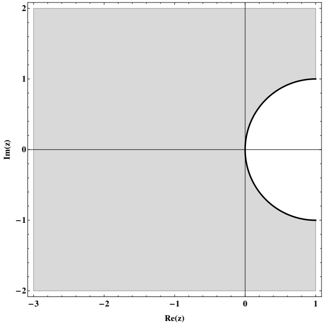


Shallow water model - NX = 201 - $\varepsilon = 10^{-6}$



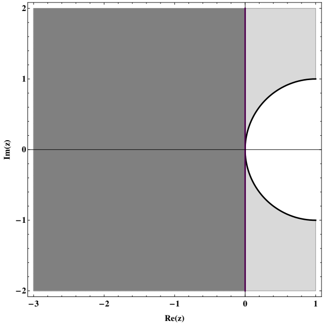
Construction of methods: Absolute stability

We require the implicit part to be *A*-stable or *L*-stable.



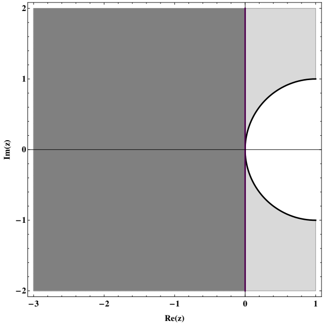
Construction of methods: Absolute stability

We require the implicit part to be *A*-stable or *L*-stable.



Construction of methods: Absolute stability

We require the implicit part to be A-stable or L-stable.



Sufficient condition based on the Schur criterion and the maximum modulus principle.

Absolute stability of the IMEX method

We apply the method to the linear test equation

$$y'(t) = \lambda_0 y(t) + \lambda_1 y(t), \quad t \geq 0,$$

Absolute stability of the IMEX method

We apply the method to the linear test equation

$$y'(t) = \lambda_0 y(t) + \lambda_1 y(t), \quad t \geq 0,$$

to obtain the recurrence relation

$$y^{[n+1]} = \mathbf{M}(z_0, z_1) y^{[n]}, \quad n = 0, 1, \ldots,$$

Absolute stability of the IMEX method

We apply the method to the linear test equation

$$y'(t) = \lambda_0 y(t) + \lambda_1 y(t), \quad t \geq 0,$$

to obtain the recurrence relation

$$y^{[n+1]} = \mathbf{M}(z_0, z_1) y^{[n]}, \quad n = 0, 1, \ldots,$$

where $z_0 = h\lambda_0$, $z_1 = h\lambda_1$, and the stability matrix $\mathbf{M}(z_0, z_1)$ is defined by

$$\mathbf{M}(z_0, z_1) = \mathbf{V} + (z_0 \mathbf{B} + z_1 \mathbf{B}^*)(\mathbf{I} - z_0 \mathbf{A} - z_1 \mathbf{A}^*)^{-1} \mathbf{U}.$$

Absolute stability of the IMEX method

We apply the method to the linear test equation

$$y'(t) = \lambda_0 y(t) + \lambda_1 y(t), \quad t \geq 0,$$

to obtain the recurrence relation

$$y^{[n+1]} = \mathbf{M}(z_0, z_1) y^{[n]}, \quad n = 0, 1, \ldots,$$

where $z_0 = h\lambda_0$, $z_1 = h\lambda_1$, and the stability matrix $\mathbf{M}(z_0, z_1)$ is defined by

$$\mathbf{M}(z_0, z_1) = \mathbf{V} + (z_0\mathbf{B} + z_1\mathbf{B}^*)(\mathbf{I} - z_0\mathbf{A} - z_1\mathbf{A}^*)^{-1}\mathbf{U}.$$

We also define the stability function $p(w, z_0, z_1)$ of the IMEX scheme as the characteristic polynomial of $\mathbf{M}(z_0, z_1)$, i.e.,

$$p(w, z_0, z_1) = \det\left(w\mathbf{I} - \mathbf{M}(z_0, z_1)\right).$$

To investigate such methods we consider the sets

$$\mathcal{S}_\alpha = \Big\{ z_0 \in \mathbb{C} : \text{ the IMEX GLM is stable for any } z_1 \in \mathcal{A}_\alpha \Big\},$$

where the set $\mathcal{A}_\alpha \subset \mathbb{C}$ is defined by

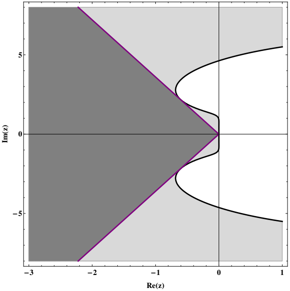
$$\mathcal{A}_\alpha = \Big\{ z \in \mathbb{C} : \text{ Re}(z) < 0 \quad \text{and} \quad |\text{Im}(z)| \leq \tan(\alpha)|\text{Re}(z)| \Big\}.$$

To investigate such methods we consider the sets

$$\mathcal{S}_\alpha = \Big\{ z_0 \in \mathbb{C} : \text{ the IMEX GLM is stable for any } z_1 \in \mathcal{A}_\alpha \Big\},$$

where the set $\mathcal{A}_\alpha \subset \mathbb{C}$ is defined by

$$\mathcal{A}_\alpha = \Big\{ z \in \mathbb{C} : \ \mathrm{Re}(z) < 0 \quad \text{and} \quad |\mathrm{Im}(z)| \leq \tan(\alpha) |\mathrm{Re}(z)| \Big\}.$$



To investigate such methods we consider the sets

$$\mathcal{S}_\alpha = \Big\{ z_0 \in \mathbb{C} : \text{ the IMEX GLM is stable for any } z_1 \in \mathcal{A}_\alpha \Big\},$$

where the set $\mathcal{A}_\alpha \subset \mathbb{C}$ is defined by

$$\mathcal{A}_\alpha = \Big\{ z \in \mathbb{C} : \text{ Re}(z) < 0 \quad \text{and} \quad |\text{Im}(z)| \leq \tan(\alpha)|\text{Re}(z)| \Big\}.$$

It follows from the maximum modulus principle that $\mathcal{S}_\alpha$ has a simple representation given by

$$\mathcal{S}_\alpha = \left\{ \begin{array}{ll} z_0 \in \mathbb{C} : & \text{the IMEX GLM is stable for any} \\ & z_1 = -|y|/\tan(\alpha) + iy, \; y \in \mathbb{R} \end{array} \right\}.$$

For fixed values of $y \in \mathbb{R}$ we define also the sets

$$\mathcal{S}_{\alpha,y} = \left\{ \begin{array}{ll} z_0 \in \mathbb{C} : & \text{the IMEX GLM is stable for fixed} \\ & z_1 = -|y|/\tan(\alpha) + iy \end{array} \right\}.$$

For fixed values of $y \in \mathbb{R}$ we define also the sets

$$\mathcal{S}_{\alpha,y} = \left\{ \begin{array}{ll} z_0 \in \mathbb{C} : & \text{the IMEX GLM is stable for fixed} \\ & z_1 = -|y|/\tan(\alpha) + iy \end{array} \right\}.$$

Observe that

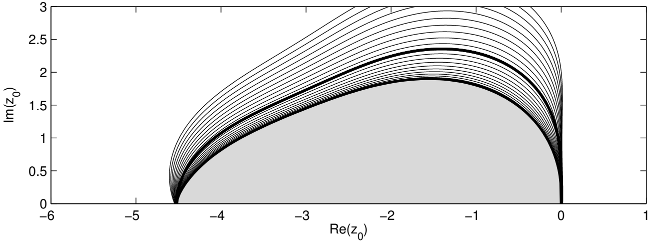
$$\mathcal{S}_\alpha = \bigcap_{y \in \mathbb{R}} \mathcal{S}_{\alpha,y}.$$

For fixed values of $y \in \mathbb{R}$ we define also the sets

$$\mathcal{S}_{\alpha,y} = \left\{ z_0 \in \mathbb{C} : \begin{array}{l} \text{the IMEX GLM is stable for fixed} \\ z_1 = -|y|/\tan(\alpha) + iy \end{array} \right\}.$$

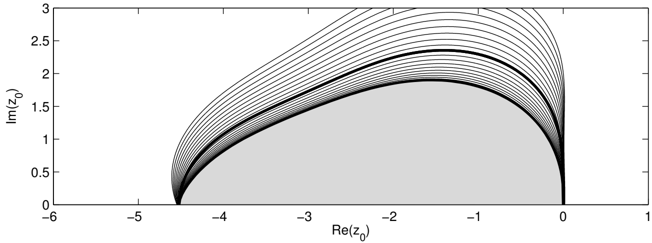
Observe that

$$\mathcal{S}_\alpha = \bigcap_{y \in \mathbb{R}} \mathcal{S}_{\alpha,y}.$$



Observe also that the region $\mathcal{S}_{\alpha,0}$ is independent of α, and corresponds to the stability region $\mathcal{S}_E$ of the explicit method, $\mathcal{S}_E$.
We have

$$\mathcal{S}_\alpha \subset \mathcal{S}_E,$$



We will look for IMEX GLMs for which the stability region $\mathcal{S}_\alpha$ contains a large part of the stability region $\mathcal{S}_E$ of the explicit method.

Local Discretization Error

- $\eta(t_n, h) \longleftarrow$ Local Error for Stages $Y^{[n]}$
- $\xi(t_n, h) \longleftarrow$ Local Error for external approximations $y^{[n]}$

Local Discretization Error

- $\eta(t_n, h) \longleftarrow$ Local Error for Stages $Y^{[n]}$
- $\xi(t_n, h) \longleftarrow$ Local Error for external approximations $y^{[n]}$

Since

$$Y^{[n+1]} = y(t_n + \mathbf{c}h) + O(h^{p+1}),$$

$$y^{[n]} = (\mathbf{W} \otimes \mathbf{I})x(h, t_n) + (\widehat{\mathbf{W}} \otimes \mathbf{I})z(h, t_n) + O(h^{p+1}),$$

and

$$y^{[n+1]} = (\mathbf{W} \otimes \mathbf{I})x(h, t_{n+1}) + (\widehat{\mathbf{W}} \otimes \mathbf{I})z(h, t_{n+1}) + O(h^{p+1}),$$

Local Discretization Error

- $\eta(t_n, h) \longleftarrow$ Local Error for stages $Y^{[n+1]}$
- $\xi(t_n, h) \longleftarrow$ Local Error for external approximations $y^{[n+1]}$

For a scalar problem we have

$$\begin{aligned}\eta(t_n, h) &= x(t_n + \mathbf{c}h) - h\mathbf{A}x'(t_n + \mathbf{c}h) - \mathbf{U}\mathbf{W}x(h, t_n) + \\ &\quad z(t_n + \mathbf{c}h) - h\widehat{\mathbf{A}}z'(t_n + \mathbf{c}h) - \mathbf{U}\widehat{\mathbf{W}}z(h, t_n) + O(h^{p+1})\end{aligned}$$

and

$$\begin{aligned}\xi(t_n, h) &= x(t_n + \mathbf{c}h) - h\mathbf{B}x'(t_n + \mathbf{c}h) - \mathbf{V}\mathbf{W}x(h, t_n) + \\ &\quad z(t_n + \mathbf{c}h) - h\widehat{\mathbf{B}}z'(t_n + \mathbf{c}h) - \mathbf{V}\widehat{\mathbf{W}}z(h, t_n) + O(h^{p+1})\end{aligned}$$

Local Discretization Error

Theorem

Assume that IMEX GLM (2) has order p and stage order $q = p$. Then the local discretization errors $\xi(t_n, h)$ and $\eta(t_n, h)$ are given by

$$\xi(t_n, h) = \psi_{p+1} h^{p+1} x^{(p+1)}(t_n) + \widehat{\psi}_{p+1} h^{p+1} z^{(p+1)}(t_n) + O(h^{p+2}),$$

$$\eta(t_n, h) = \varphi_{p+1} h^{p+1} x^{(p+1)}(t_n) + \widehat{\varphi}_{p+1} h^{p+1} z^{(p+1)}(t_n) + O(h^{p+2}),$$

Local Discretization Error

Theorem

Assume that IMEX GLM (2) has order p and stage order $q = p$. Then the local discretization errors $\xi(t_n, h)$ and $\eta(t_n, h)$ are given by

$$\xi(t_n, h) = \psi_{p+1} h^{p+1} x^{(p+1)}(t_n) + \widehat{\psi}_{p+1} h^{p+1} z^{(p+1)}(t_n) + O(h^{p+2}),$$

$$\eta(t_n, h) = \varphi_{p+1} h^{p+1} x^{(p+1)}(t_n) + \widehat{\varphi}_{p+1} h^{p+1} z^{(p+1)}(t_n) + O(h^{p+2}),$$

where ψ_{p+1}, $\widehat{\psi}_{p+1}$, φ_{p+1}, and $\widehat{\varphi}_{p+1}$, are given by

$$\psi_{p+1} = \sum_{k=0}^{p} \frac{\mathbf{q}_k}{(p+1-k)!} - \mathbf{B}\frac{\mathbf{c}^p}{p!}, \quad \widehat{\psi}_{p+1} = \sum_{k=0}^{p} \frac{\widehat{\mathbf{q}}_k}{(p+1-k)!} - \widehat{\mathbf{B}}\frac{\mathbf{c}^p}{p!}.$$

$$\varphi_{p+1} = \frac{\mathbf{c}^{p+1}}{(p+1)!} - \mathbf{A}\frac{\mathbf{c}^p}{p!}, \quad \widehat{\varphi}_{p+1} = \frac{\mathbf{c}^{p+1}}{(p+1)!} - \widehat{\mathbf{A}}\frac{\mathbf{c}^p}{p!}.$$

Theorem

Assume that IMEX GLM (2) has order p and stage order $q = p$. Then

$$h^{p+1}x^{(p+1)}(t_n) = h\sum_{j=1}^{s}\left(\alpha_j f\big(Y_j^{[n+1]}\big) + \beta_j f\big(Y_j^{[n]}\big)\right) + O(h^{p+2})$$

and

$$h^{p+1}z^{(p+1)}(t_n) = h\sum_{j=1}^{s}\left(\alpha_j\, g\big(Y_j^{[n+1]}\big) + \beta_j\, g\big(Y_j^{[n]}\big)\right) + O(h^{p+2}),$$

Theorem

Assume that IMEX GLM (2) has order p and stage order q = p. Then

$$h^{p+1}x^{(p+1)}(t_n) = h\sum_{j=1}^{s}\Big(\alpha_j\, f\big(Y_j^{[n+1]}\big) + \beta_j\, f\big(Y_j^{[n]}\big)\Big) + O(h^{p+2})$$

and

$$h^{p+1}z^{(p+1)}(t_n) = h\sum_{j=1}^{s}\Big(\alpha_j\, g\big(Y_j^{[n+1]}\big) + \beta_j\, g\big(Y_j^{[n]}\big)\Big) + O(h^{p+2}),$$

Theorem

Assume that IMEX GLM (2) has order p and stage order q = p. Then

$$h^{p+1}x^{(p+1)}(t_n) = h\sum_{j=1}^{s}\left(\alpha_j\, f\left(Y_j^{[n+1]}\right) + \beta_j\, f\left(Y_j^{[n]}\right)\right) + O(h^{p+2})$$

and

$$h^{p+1}z^{(p+1)}(t_n) = h\sum_{j=1}^{s}\left(\alpha_j\, g\left(Y_j^{[n+1]}\right) + \beta_j\, g\left(Y_j^{[n]}\right)\right) + O(h^{p+2}),$$

where α_j *and* β_j, $j = 1, 2, \ldots, s$, *satisfy*

$$\begin{cases} \displaystyle\sum_{j=1}^{s}\left(\alpha_j c_j^{k-1} + \beta_j (c_j - 1)^{k-1}\right) = 0, \quad k = 1, 2, \ldots, p, \\ \displaystyle\sum_{j=1}^{s}\left(\frac{\alpha_j c_j^{p}}{p!} + \frac{\beta_j (c_j - 1)^{p}}{p!}\right) = 1. \end{cases}$$

IMEX GLMs on Nonuniform Grid

$$t_0 < t_1 < \cdots < t_N, \quad t_N \geq T, \quad h_n = t_{n+1} - t_n, \quad n = 0, 1, \ldots, N-1,$$

IMEX GLMs on Nonuniform Grid

$$t_0 < t_1 < \cdots < t_N, \quad t_N \geq T, \quad h_n = t_{n+1} - t_n, \quad n = 0, 1, \ldots, N-1,$$

$$\begin{cases} Y^{[n+1]} = h_n(\mathbf{A} \otimes \mathbf{I})f\left(Y^{[n+1]}\right) + h_n(\widehat{\mathbf{A}} \otimes \mathbf{I})g\left(Y^{[n+1]}\right) + (\mathbf{U} \otimes \mathbf{I})\overline{y}^{[n]}, \\ y^{[n+1]} = h_n(\mathbf{B} \otimes \mathbf{I})f\left(Y^{[n+1]}\right) + h_n(\widehat{\mathbf{B}} \otimes \mathbf{I})g\left(Y^{[n+1]}\right) + (\mathbf{V} \otimes \mathbf{I})\overline{y}^{[n]}, \end{cases} \tag{3}$$

IMEX GLMs on Nonuniform Grid

$$t_0 < t_1 < \cdots < t_N, \quad t_N \geq T, \quad h_n = t_{n+1} - t_n, \quad n = 0, 1, \ldots, N-1,$$

$$\begin{cases} Y^{[n+1]} = h_n(\mathbf{A} \otimes \mathbf{I}) f\left(Y^{[n+1]}\right) + h_n(\widehat{\mathbf{A}} \otimes \mathbf{I}) g\left(Y^{[n+1]}\right) + (\mathbf{U} \otimes \mathbf{I}) \overline{y}^{[n]}, \\ y^{[n+1]} = h_n(\mathbf{B} \otimes \mathbf{I}) f\left(Y^{[n+1]}\right) + h_n(\widehat{\mathbf{B}} \otimes \mathbf{I}) g\left(Y^{[n+1]}\right) + (\mathbf{V} \otimes \mathbf{I}) \overline{y}^{[n]}, \end{cases} \tag{3}$$

where

$$Y^{[n+1]} = y(t_n + \mathbf{c}h_n) + O(h^{p+1}),$$

$$\overline{y}^{[n]} = (\mathbf{W} \otimes \mathbf{I}) x(h_n, t_n) + (\widehat{\mathbf{W}} \otimes \mathbf{I}) z(h_n, t_n) + O(h^{p+1}),$$

IMEX GLMs on Nonuniform Grid

$$t_0 < t_1 < \cdots < t_N, \quad t_N \geq T, \quad h_n = t_{n+1} - t_n, \quad n = 0, 1, \ldots, N-1,$$

$$\begin{cases} Y^{[n+1]} = h_n(\mathbf{A} \otimes \mathbf{I}) f\big(Y^{[n+1]}\big) + h_n(\widehat{\mathbf{A}} \otimes \mathbf{I}) g\big(Y^{[n+1]}\big) + (\mathbf{U} \otimes \mathbf{I})\overline{y}^{[n]}, \\ y^{[n+1]} = h_n(\mathbf{B} \otimes \mathbf{I}) f\big(Y^{[n+1]}\big) + h_n(\widehat{\mathbf{B}} \otimes \mathbf{I}) g\big(Y^{[n+1]}\big) + (\mathbf{V} \otimes \mathbf{I})\overline{y}^{[n]}, \end{cases} \quad (3)$$

where

$$Y^{[n+1]} = y(t_n + \mathbf{c}h_n) + O(h^{p+1}),$$

$$\overline{y}^{[n]} = (\mathbf{W} \otimes \mathbf{I})x(h_n, t_n) + (\widehat{\mathbf{W}} \otimes \mathbf{I})z(h_n, t_n) + O(h^{p+1}),$$

$$y^{[n+1]} = (\mathbf{W} \otimes \mathbf{I})x(h_n, t_{n+1}) + (\widehat{\mathbf{W}} \otimes \mathbf{I})z(h_n, t_{n+1}) + O(h^{p+1}),$$

and $h = \max\{h_n : n = 0, 1, \ldots, N\}$.

Theorem

Assume that IMEX GLM (3) has order p and stage order q = p. Then

$$h_n^{p+1} x^{(p+1)}(t_n) = h_n \sum_{j=1}^{s} \left(\alpha_j f\left(Y_j^{[n+1]}\right) + \beta_j f\left(Y_j^{[n]}\right) \right) + O(h^{p+2})$$

and

$$h_n^{p+1} z^{(p+1)}(t_n) = h_n \sum_{j=1}^{s} \left(\alpha_j g\left(Y_j^{[n+1]}\right) + \beta_j g\left(Y_j^{[n]}\right) \right) + O(h^{p+2}),$$

Theorem

Assume that IMEX GLM (3) has order p and stage order q = p. Then

$$h_n^{p+1} x^{(p+1)}(t_n) = h_n \sum_{j=1}^{s} \left(\alpha_j f\left(Y_j^{[n+1]}\right) + \beta_j f\left(Y_j^{[n]}\right) \right) + O(h^{p+2})$$

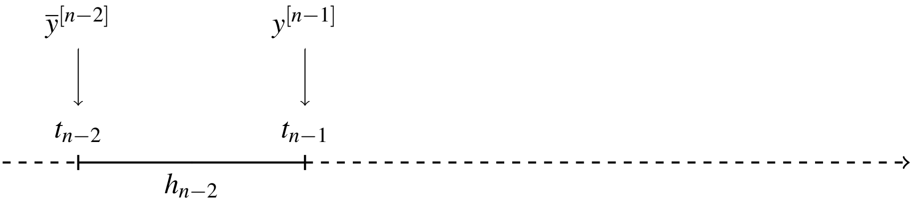
and

$$h_n^{p+1} z^{(p+1)}(t_n) = h_n \sum_{j=1}^{s} \left(\alpha_j g\left(Y_j^{[n+1]}\right) + \beta_j g\left(Y_j^{[n]}\right) \right) + O(h^{p+2}),$$

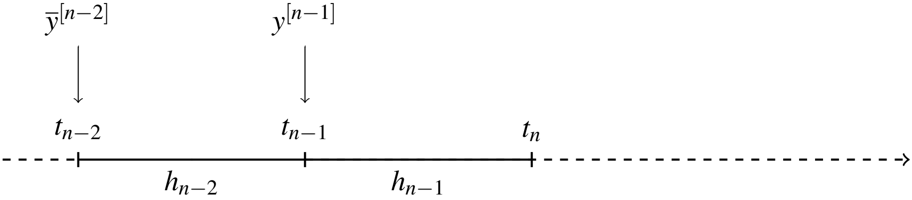
where $\delta_n = h_n / h_{n-1}$, *and* $\alpha_j = \alpha_j(\delta_n)$ *and* $\beta_j = \beta_j(\delta_n)$, $j = 1, 2, \ldots, s$, *satisfy*

$$\begin{cases} \displaystyle\sum_{j=1}^{s} \left(\alpha_j c_j^{k-1} + \frac{\beta_j (c_j - 1)^{k-1}}{\delta_n^{k-1}} \right) = 0, \quad k = 1, 2, \ldots, p, \\ \displaystyle\sum_{j=1}^{s} \left(\frac{\alpha_j c_j^p}{p!} + \frac{\beta_j (c_j - 1)^p}{p!\, \delta_n^p} \right) = 1. \end{cases}$$

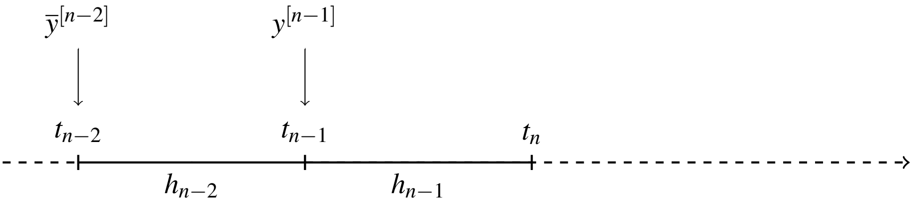
Changing the stepsize $h_{n-1} \longrightarrow h_n$



Changing the stepsize $h_{n-1} \longrightarrow h_n$



Changing the stepsize $h_{n-1} \longrightarrow h_n$

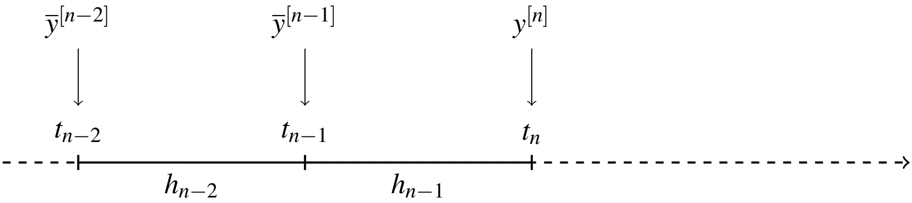


$$h_{n-1} = h_{n-2}$$

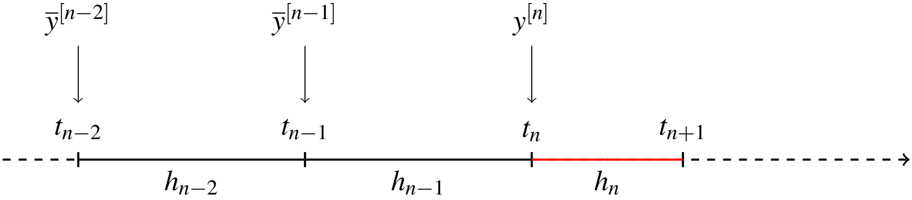
$$\Downarrow$$

$y^{[n-1]} = \bar{y}^{[n-1]}$, ✓ we can compute $y^{[n]}$

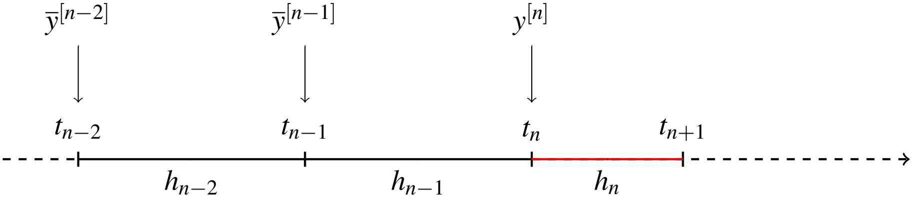
Changing the stepsize $h_{n-1} \longrightarrow h_n$



Changing the stepsize $h_{n-1} \longrightarrow h_n$



Changing the stepsize $h_{n-1} \longrightarrow h_n$



$$h_n \neq h_{n-1}$$

$$\Downarrow$$

✗ **We need to rescale $y^{[n]}$ to $\bar{y}^{[n]}$ in order to compute $y^{[n+1]}$**

Rescaling the vectors of external approximations

Changing the stepsize

$$h_{n-1} \longrightarrow h_n$$

we need to compute the vector of external approximations $\bar{y}^{[n]}$ corresponding to the new stepsize h_n

$$\bar{y}^{[n]} = (\mathbf{W} \otimes \mathbf{I})x(h_n, t_n) + (\widehat{\mathbf{W}} \otimes \mathbf{I})z(h_n, t_n) + O(h^{p+1}),$$

in terms of the vector $y^{[n]}$ corresponding to h_{n-1}, which satisfies

$$y^{[n]} = (\mathbf{W} \otimes \mathbf{I})x(h_{n-1}, t_n) + (\widehat{\mathbf{W}} \otimes \mathbf{I})z(h_{n-1}, t_n) + O(h^{p+1}).$$

Let us define the *stepsize ratio*

$$\delta_n = h_n / h_{n-1}$$

and the matrix

$$\mathbf{D}(\delta) = \mathrm{diag}\left(\left[\begin{array}{cccc} 1, & \delta, & \cdots, & \delta^p \end{array} \right] \right).$$

Let us define the *stepsize ratio*

$$\delta_n = h_n/h_{n-1}$$

and the matrix

$$\mathbf{D}(\delta) = \operatorname{diag}\left(\begin{bmatrix} 1, & \delta, & \cdots, & \delta^p \end{bmatrix}\right).$$

Since

$$x(h,t) = \begin{bmatrix} x(t) \\ hx'(t) \\ \vdots \\ h^p x^{(p)}(t) \end{bmatrix}, \quad z(h,t) = \begin{bmatrix} z(t) \\ hz'(t) \\ \vdots \\ h^p z^{(p)}(t) \end{bmatrix},$$

it holds

$$x(h_n,t_n) = \mathbf{D}(\delta)x(h_{n-1},t_n) \quad \text{and} \quad z(h_n,t_n) = \mathbf{D}(\delta)z(h_{n-1},t_n).$$

The vectors $\overline{y}^{[n]}$ and $y^{[n]}$ satisfy the relations

$$y^{[n]} = (\mathbf{W} \otimes \mathbf{I})x(h_{n-1}, t_n) + (\widehat{\mathbf{W}} \otimes \mathbf{I})z(h_{n-1}, t_n) + O(h^{p+1}),$$

and

$$\overline{y}^{[n]} = (\mathbf{W} \otimes \mathbf{I})x(h_n, t_n) + (\widehat{\mathbf{W}} \otimes \mathbf{I})z(h_n, t_n) + O(h^{p+1}),$$

So

$$\overline{y}^{[n]} = (\mathbf{W} \otimes \mathbf{I})\mathbf{D}(\delta)x(h_{n-1}, t_n) + (\widehat{\mathbf{W}} \otimes \mathbf{I})\mathbf{D}(\delta)z(h_{n-1}, t_n) + O(h^{p+1}),$$

Strategy #1 - Assuming $\widehat{\mathbf{W}} = I$

Taking this into account $\bar{y}^{[n]}$ and $y^{[n]}$ can be written in the form

$$\bar{y}^{[n]} = \big(\mathbf{W}\mathbf{D}(\delta_n)\mathbf{W}^{-1}\otimes\mathbf{I}\big)\big(\mathbf{W}\otimes\mathbf{I}\big)x(h_{n-1},t_n)+\big(\mathbf{D}(\delta_n)\otimes\mathbf{I}\big)z(h_{n-1},t_n)+O(h^{p+1})$$

$$y^{[n]} = (\mathbf{W}\otimes\mathbf{I})x(h_{n-1},t_n) + z(h_{n-1},t_n) + O(h^{p+1}).$$

It follows that

$$\bar{y}^{[n]} = \big(\mathbf{D}(\delta_n)\otimes\mathbf{I}\big)y^{[n]} + O(h^{p+1})$$

if

$$\mathbf{W}\,\mathbf{D}(\delta_n)\mathbf{W}^{-1} = \mathbf{D}(\delta_n).$$

This is clearly the case if $\mathbf{W}$ is diagonal.

Strategy #1 - Assuming $\widehat{\mathbf{W}} = I$

Taking this into account $\bar{y}^{[n]}$ and $y^{[n]}$ can be written in the form

$$\bar{y}^{[n]} = \left(\mathbf{W}\mathbf{D}(\delta_n)\mathbf{W}^{-1}\otimes\mathbf{I}\right)\left(\mathbf{W}\otimes\mathbf{I}\right)x(h_{n-1},t_n)+\left(\mathbf{D}(\delta_n)\otimes\mathbf{I}\right)z(h_{n-1},t_n)+O(h^{p+1})$$

$$y^{[n]} = (\mathbf{W}\otimes\mathbf{I})x(h_{n-1},t_n) + z(h_{n-1},t_n) + O(h^{p+1}).$$

It follows that

$$\bar{y}^{[n]} = \left(\mathbf{D}(\delta_n)\otimes\mathbf{I}\right)y^{[n]} + O(h^{p+1})$$

if

$$\mathbf{W}\,\mathbf{D}(\delta_n)\mathbf{W}^{-1} = \mathbf{D}(\delta_n).$$

This is clearly the case if $\mathbf{W}$ is diagonal.
Unfortunately this is possible only for methods of order $p \leq 2$.

Strategy #2 - Rescaling $x^{[n]}$ and $z^{[n]}$

We consider

$$y^{[n]} = (\mathbf{W} \otimes \mathbf{I})x(h_{n-1}, t_n) + (\widehat{\mathbf{W}} \otimes \mathbf{I})z(h_{n-1}, t_n) + O(h^{p+1}),$$

as

$$y^{[n]} = x^{[n]} + z^{[n]} + O(h^{p+1})$$

we can rescale the vectors

$$x^{[n]} = (\mathbf{W} \otimes \mathbf{I})x(h_{n-1}, t_n) + O(h^{p+1}),$$

$$z^{[n]} = (\widehat{\mathbf{W}} \otimes \mathbf{I})z(h_{n-1}, t_n) + O(h^{p+1}).$$

which correspond to the old stepsize h_{n-1}, respectively to $\overline{x}^{[n]}$ and $\overline{z}^{[n]}$ which satisfy

$$\overline{x}^{[n]} = (\mathbf{W} \otimes \mathbf{I})x(h_n, t_n) + O(h^{p+1}),$$

$$\overline{z}^{[n]} = (\widehat{\mathbf{W}} \otimes \mathbf{I})z(h_n, t_n) + O(h^{p+1}),$$

and correspond to the new stepsize h_n.

Strategy #2 - Rescaling $x^{[n]}$ and $z^{[n]}$

Consider

$$\begin{aligned}\bar{x}^{[n]} &= \big(\mathbf{W}\mathbf{D}(\delta_n) \otimes \mathbf{I}\big) x(h_{n-1}, t_n) + O(h^{p+1}) \\ &= \big(\mathbf{W}\mathbf{D}(\delta_n)\mathbf{W}^{-1} \otimes \mathbf{I}\big)(\mathbf{W} \otimes \mathbf{I}) x(h_{n-1}, t_n) + O(h^{p+1}).\end{aligned}$$

So we have

$$\bar{x}^{[n]} = \big(\mathbf{W}\mathbf{D}(\delta_n)\mathbf{W}^{-1} \otimes \mathbf{I}\big) x^{[n]},$$

and similarly we obtain

$$\bar{z}^{[n]} = \big(\widehat{\mathbf{W}}\mathbf{D}(\delta_n)\widehat{\mathbf{W}}^{-1} \otimes \mathbf{I}\big) z^{[n]}.$$

Strategy #2 - Rescaling $x^{[n]}$ and $z^{[n]}$

Consider

$$\begin{aligned}\overline{x}^{[n]} &= \big(\mathbf{W}\mathbf{D}(\delta_n)\otimes \mathbf{I}\big)x(h_{n-1},t_n)+O(h^{p+1})\\ &= \big(\mathbf{W}\mathbf{D}(\delta_n)\mathbf{W}^{-1}\otimes \mathbf{I}\big)\big(\mathbf{W}\otimes \mathbf{I}\big)x(h_{n-1},t_n)+O(h^{p+1}).\end{aligned}$$

So we have

$$\overline{x}^{[n]} = \big(\mathbf{W}\mathbf{D}(\delta_n)\mathbf{W}^{-1}\otimes \mathbf{I}\big)x^{[n]},$$

and similarly we obtain

$$\overline{z}^{[n]} = \big(\widehat{\mathbf{W}}\mathbf{D}(\delta_n)\widehat{\mathbf{W}}^{-1}\otimes \mathbf{I}\big)z^{[n]}.$$

Hence

$$\overline{y}^{[n]} = \overline{x}^{[n]} + \overline{z}^{[n]}.$$

Ingredients for construction of methods

- Stability Properties
  (e.g. A-stability for the implicit part,
  Strong Stability Preservation Property,
  Overall linear stability, ...)

Ingredients for construction of methods

- Stability Properties
  (e.g. A-stability for the implicit part,
  Strong Stability Preservation Property,
  Overall linear stability, ...)

- Starting Procedure

- Finishing Procedure
  (e.g. $Y_s^{[n+1]} = y(t_n + h) + O(h^{p+1})$)

- Implementation strategies

Ingredients for construction of methods

- Stability Properties
  (e.g. A-stability for the implicit part,
  Strong Stability Preservation Property,
  Overall linear stability, ...)

- Starting Procedure

- Finishing Procedure
  (e.g. $Y_s^{[n+1]} = y(t_n + h) + O(h^{p+1})$)

- Implementation strategies

We have not focused on construction of methods here, because this approach is valid for several classes of GLMs.

Numerical Experiments

Problem 1 - Linear Test Equation

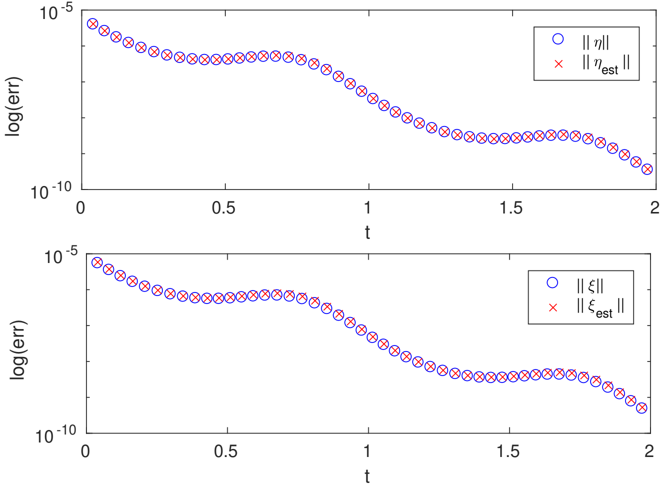
$$\begin{cases} y'(t) &= \lambda_0\, y(t) + \lambda_1\, y(t), \\ y(t_0) &= y_0, \end{cases}$$

with $\lambda_0 = -1$, $\lambda_1 = -15$, $y_0 = 1$, $T = 2$, $N = 200$.

- Variable stepsizes, chosen according to the formula

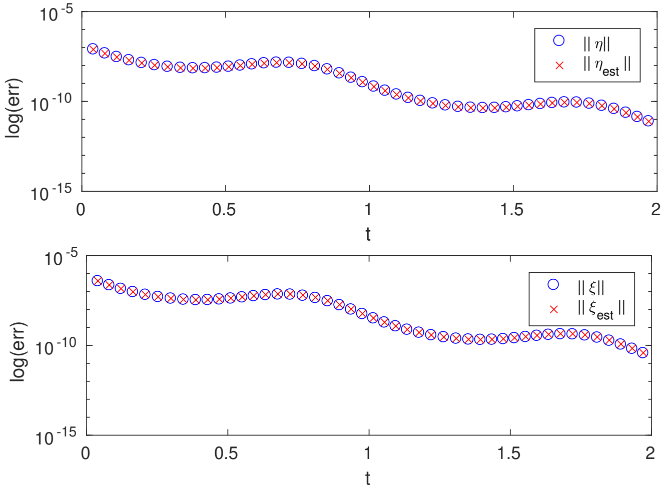
$$h_n = \rho^{(-1)^n \sin(4\pi t_n/(T-t_0))}\, h_{n-1}, \quad n = 1, 2, \ldots, N,$$

Problem 1 - Linear Test Equation, Variable stepsize



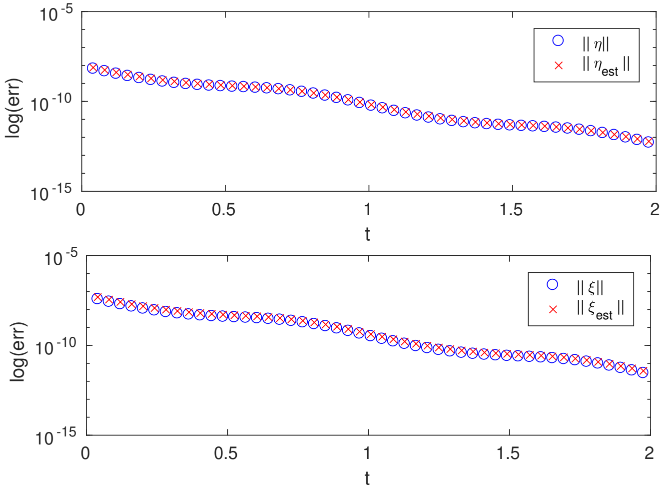
LEE for IMEX GLM of order $p = 2$

Problem 1 - Linear Test Equation, Variable stepsize



LEE for IMEX GLM of order $p = 3$

Problem 1 - Linear Test Equation, Variable stepsize



LEE for IMEX GLM of order $p = 4$

Numerical Experiments

Problem 2 - Prothero-Robinson Equation

$$\begin{cases} y'(t) = -\mu(y(t) - \phi(t)) + \phi'(t), \\ y(0) = \phi(0), \end{cases}$$

with $\mu = -10$, $\phi(t) = sin(t)$, $T = 2$, $N = 200$.

Numerical Experiments

Problem 2 - Prothero-Robinson Equation

$$\begin{cases} y'(t) &= -\mu(y(t) - \phi(t)) + \phi'(t), \\ y(0) &= \phi(0), \end{cases}$$

with $\mu = -10$, $\phi(t) = sin(t)$, $T = 2$, $N = 200$.

- Constant stepsizes, $h_n = h = const$,

Numerical Experiments

Problem 2 - Prothero-Robinson Equation

$$\begin{cases} y'(t) &= -\mu(y(t)-\phi(t))+\phi'(t), \\ y(0) &= \phi(0), \end{cases}$$

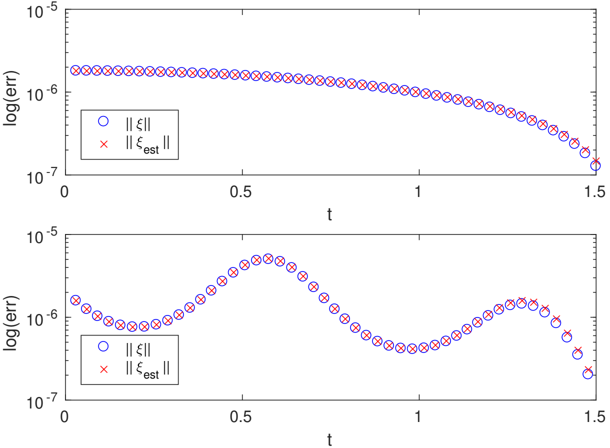
with $\mu = -10$, $\phi(t) = sin(t)$, $T = 2$, $N = 200$.

- Constant stepsizes, $h_n = h = const$,
- Variable stepsizes, chosen according to the formula

$$h_n = \rho^{(-1)^n \sin(4\pi t_n/(T-t_0))} h_{n-1}, \quad n = 1, 2, \ldots, N,$$

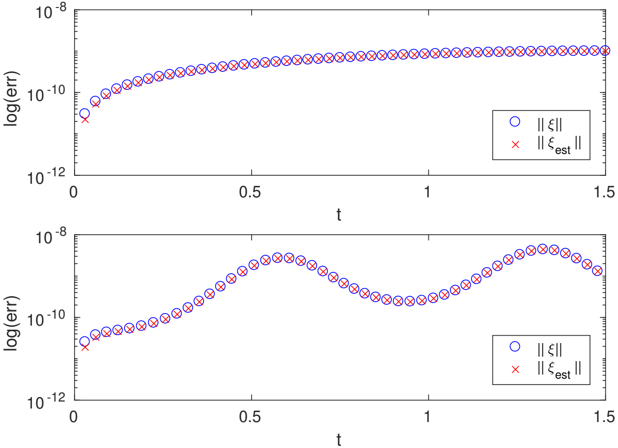
$\rho = 2$ for methods of order $p = 2$, $p = 3$, and
$\rho = 1.25$ for methods of order $p = 4$.

Problem 2 - Prothero-Robinson Equation



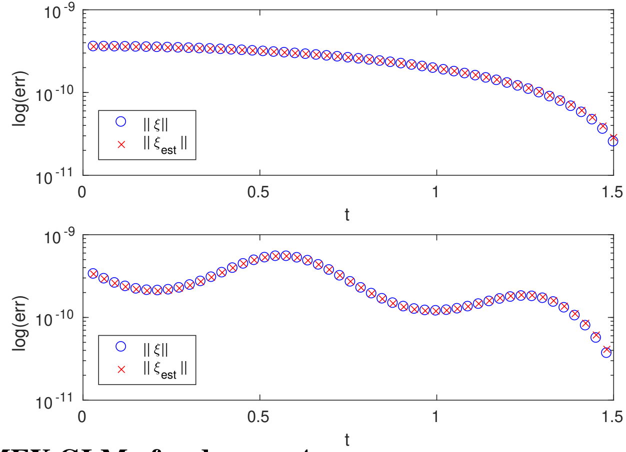
LEE for IMEX GLM of order $p = 2$,
Fixed stepsize (top), Variable stepsize $\rho = 2$ (bottom)

Problem 2 - Prothero-Robinson Equation



LEE for IMEX GLM of order $p = 3$,
Fixed stepsize (top), Variable stepsize $\rho = 2$ (bottom)

Problem 2 - Prothero-Robinson Equation



LEE for IMEX GLM of order $p = 4$,
Fixed stepsize (top), Variable stepsize $\rho = 1.25$ (bottom)

VS Implementation. Example : Shallow water model

$$\begin{cases} \dfrac{\partial}{\partial t}h + \dfrac{\partial}{\partial x}(hv) = 0, \\ \dfrac{\partial}{\partial t}(hv) + \dfrac{\partial}{\partial x}\left(h + \dfrac{1}{2}h^2\right) = \dfrac{1}{\varepsilon}\left(\dfrac{h^2}{2} - hv\right), \end{cases}$$

where h is the water height with respect to the bottom and hv is the flux.

We use periodic boundary conditions and initial conditions at $t_0 = 0$

$$h(0,x) = 1 + \frac{1}{5}sin(8\pi x), \quad hv(0,x) = \frac{1}{2}h(0,x)^2, \quad \text{with } x \in [0,1].$$

VS Implementation. Example : Shallow water model

$$\begin{cases} \frac{\partial}{\partial t}h + \frac{\partial}{\partial x}(hv) = 0, \\ \frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}\left(h + \frac{1}{2}h^2\right) = \frac{1}{\varepsilon}\left(\frac{h^2}{2} - hv\right), \end{cases}$$

where h is the water height with respect to the bottom and hv is the flux.

We use periodic boundary conditions and initial conditions at $t_0 = 0$

$$h(0,x) = 1 + \frac{1}{5}sin(8\pi x), \quad hv(0,x) = \frac{1}{2}h(0,x)^2, \quad \text{with } x \in [0,1].$$

The space derivative was discretized by a fifth order finite difference weighted essentially non-oscillatory (WENO5)

		IMEX GLM			ode15s		
ε	NX	steps	fevals	sec.	steps	fevals	sec.
10^{-1}	100						
10^{-1}	200						
10^{-1}	400						
10^{-2}	100						
10^{-2}	200						
10^{-2}	400						
10^{-4}	100						
10^{-4}	200						
10^{-4}	400						
10^{-6}	100						
10^{-6}	200						
10^{-6}	400						

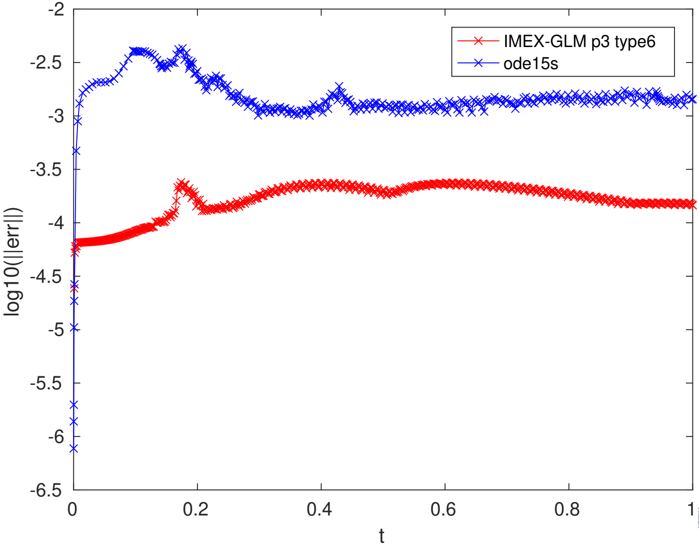
		IMEX GLM			ode15s		
ε	NX	steps	fevals	sec.	steps	fevals	sec.
10^{-1}	100	211			301		
10^{-1}	200	303			469		
10^{-1}	400	581			660		
10^{-2}	100	155			124		
10^{-2}	200	265			162		
10^{-2}	400	522			242		
10^{-4}	100	550			315		
10^{-4}	200	733			563		
10^{-4}	400	1190			994		
10^{-6}	100	1300			324		
10^{-6}	200	1350			592		
10^{-6}	400	2451			1089		

		IMEX GLM			ode15s		
ε	NX	steps	fevals	sec.	steps	fevals	sec.
10^{-1}	100		4890			19225	
10^{-1}	200		7777			52676	
10^{-1}	400		15690			188628	
10^{-2}	100		4249			4717	
10^{-2}	200		7802			18093	
10^{-2}	400		14797			38248	
10^{-4}	100		11333			12209	
10^{-4}	200		16065			50303	
10^{-4}	400		27970			185887	
10^{-6}	100		24972			11635	
10^{-6}	200		27840			45136	
10^{-6}	400		51136			171706	

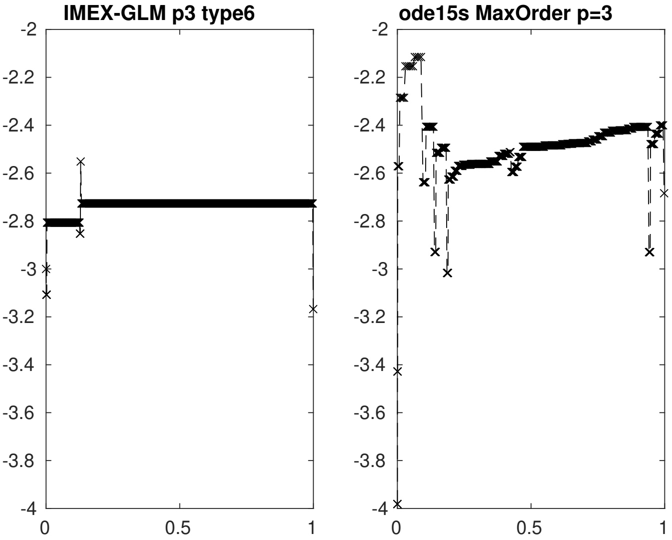
		IMEX GLM			ode15s		
ε	NX	steps	fevals	sec.	steps	fevals	sec.
10^{-1}	100			3,7			27,9
10^{-1}	200			12,5			82,7
10^{-1}	400			80,3			345,3
10^{-2}	100			2,6			7,2
10^{-2}	200			18,4			29,3
10^{-2}	400			73,9			71,1
10^{-4}	100			8,7			18,1
10^{-4}	200			33			83
10^{-4}	400			237			343
10^{-6}	100			19,7			17,4
10^{-6}	200			80,6			240
10^{-6}	400			297			327

		IMEX GLM			ode15s		
ε	NX	steps	fevals	sec.	steps	fevals	sec.
10^{-1}	100	211	4890	3,7	301	19225	27,9
10^{-1}	200	303	7777	12,5	469	52676	82,7
10^{-1}	400	581	15690	80,3	660	188628	345,3
10^{-2}	100	155	4249	2,6	124	4717	7,2
10^{-2}	200	265	7802	18,4	162	18093	29,3
10^{-2}	400	522	14797	73,9	242	38248	71,1
10^{-4}	100	550	11333	8,7	315	12209	18,1
10^{-4}	200	733	16065	33	563	50303	83
10^{-4}	400	1190	27970	237	994	185887	343
10^{-6}	100	1300	24972	19,7	324	11635	17,4
10^{-6}	200	1350	27840	80,6	592	45136	240
10^{-6}	400	2451	51136	297	1089	171706	327

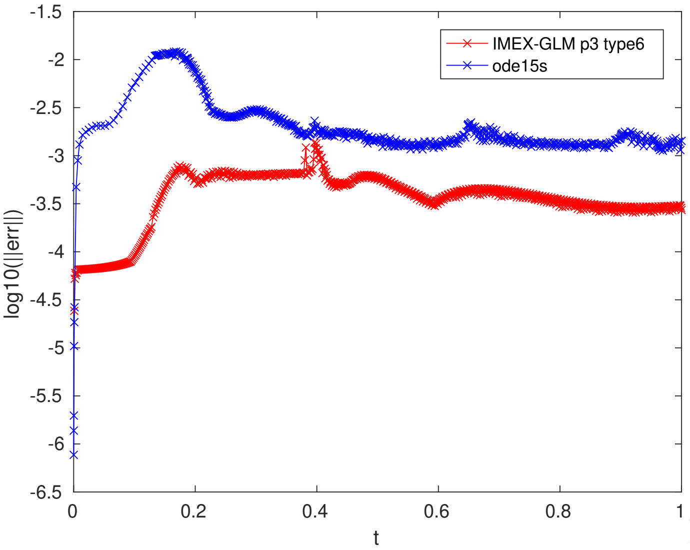
Example : Shallow water model, $\varepsilon = 10^{-4}$, $NX = 100$



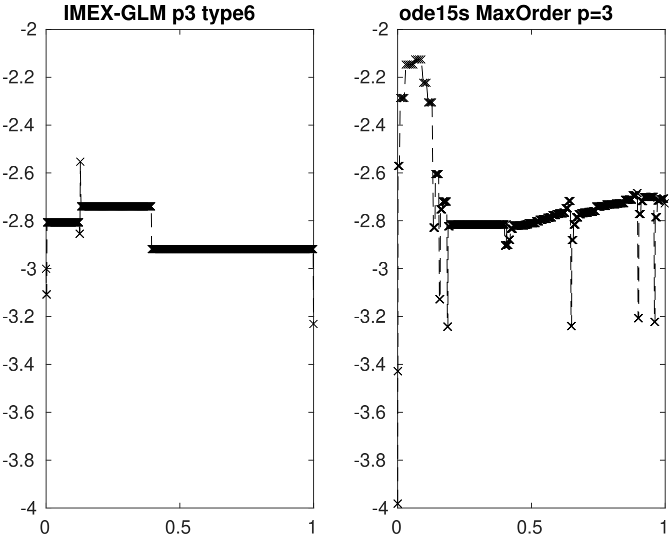
Example : Shallow water model, $\varepsilon = 10^{-4}$, $NX = 100$



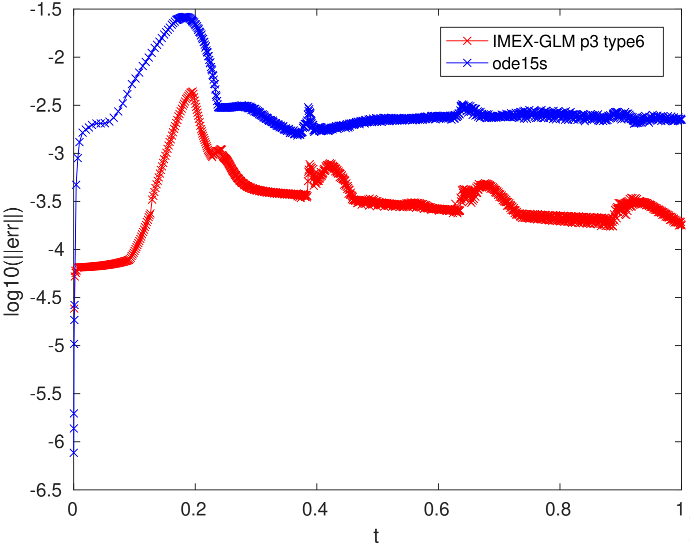
Example : Shallow water model, $\varepsilon = 10^{-4}$, $NX = 200$



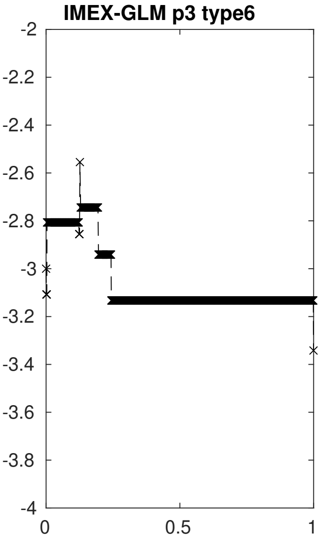
Example : Shallow water model, $\varepsilon = 10^{-4}$, $NX = 200$



Example : Shallow water model, $\varepsilon = 10^{-4}$, $NX = 400$



Example : Shallow water model, $\varepsilon = 10^{-4}$, $NX = 400$



1. Implicit-explicit General Linear Methods
   - Partitioned System
   - Order and Stage Order
2. Some Motivation
3. Local discretization error for IMEX GLMs
4. Local error estimation for IMEX GLMs
5. Nonuniform Grid
6. Rescaling the external approximations
7. Numerical Experiments
8. Concluding Remarks

Work in progress and concluding remarks

- Solver based on variable stepsize implementation of IMEX GLMs.
  - Optimized values for α and β vectors.
  - Different choices for α and β vectors (that is α, $\widehat{\alpha}$, β, $\widehat{\beta}$).
  - Construction of IMEX GLMs with *balanced* Local Errors between Implicit and Explicit part.
  - Strategies to reduce the computational cost.
- Applicable to different classes of high stage order methods, such as DIMSIMs, Generalized Linear Multistep Methods, Peer methods, ...

[1] M. Braś, G. Izzo, Z. Jackiewicz, Accurate Implicit-Explicit General Linear Methods with Inherent Runge-Kutta Stability, *J. Sci. Comput.* 70(2017),1105–1143.

[2] G. Califano, G. Izzo, and Z. Jackiewicz, Starting procedures for general linear methods, *Appl. Numer. Math*. 120(2017), 165–175.

[3] G. Califano, G. Izzo, and Z. Jackiewicz, Strong stability preserving general linear methods with Runge-Kutta stability, *J. Sci. Comput* 76(2), (2018), 943–968.

[4] G. Izzo and Z. Jackiewicz, Strong stability preserving general linear methods, *J. Sci. Comput.* 65(2015), 271–298.

[5] G. Izzo and Z. Jackiewicz, Highly stable implicit-explicit Runge-Kutta methods, *Appl. Numer. Math.*, 113(2017) 71–92.

[6] G. Izzo and Z. Jackiewicz, Strong stability preserving transformed DIMSIMs, *J. Comput. Appl. Math.* 343 (2018), 174–188.

[7] G. Izzo and Z. Jackiewicz, Transformed implicit-explicit DIMSIMs with strong stability preserving explicit part, *Numer. Alg.* 81(4), (2019), 1343–1359.

[8] G. Izzo and Z. Jackiewicz, Strong stability preserving implicit-explicit transformed general linear methods, *Math. Comput. Simul.*, (2019)

[9] G. Izzo and Z. Jackiewicz, Strong Stability Preserving IMEX Methods for Partitioned Systems of Differential Equations, *Spinger CAMC* Vol.4(3), (2021), 719–754.

Thank you for your attention!

[1] M. Braś, G. Izzo, Z. Jackiewicz, Accurate Implicit-Explicit General Linear Methods with Inherent Runge-Kutta Stability, *J. Sci. Comput.* 70(2017),1105–1143.

[2] G. Califano, G. Izzo, and Z. Jackiewicz, Starting procedures for general linear methods, *Appl. Numer. Math*. 120(2017), 165–175.

[3] G. Califano, G. Izzo, and Z. Jackiewicz, Strong stability preserving general linear methods with Runge-Kutta stability, *J. Sci. Comput* 76(2), (2018), 943–968.

[4] G. Izzo and Z. Jackiewicz, Strong stability preserving general linear methods, *J. Sci. Comput.* 65(2015), 271–298.

[5] G. Izzo and Z. Jackiewicz, Highly stable implicit-explicit Runge-Kutta methods, *Appl. Numer. Math.*, 113(2017) 71–92.

[6] G. Izzo and Z. Jackiewicz, Strong stability preserving transformed DIMSIMs, *J. Comput. Appl. Math.* 343 (2018), 174–188.

[7] G. Izzo and Z. Jackiewicz, Transformed implicit-explicit DIMSIMs with strong stability preserving explicit part, *Numer. Alg.* 81(4), (2019), 1343–1359.

[8] G. Izzo and Z. Jackiewicz, Strong stability preserving implicit-explicit transformed general linear methods, *Math. Comput. Simul.*, (2019)

[9] G. Izzo and Z. Jackiewicz, Strong Stability Preserving IMEX Methods for Partitioned Systems of Differential Equations, *Spinger CAMC* Vol.4(3), (2021), 719–754.