Recent Advances in Computational Shape Optimization

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joint work with Stephan Schmidt, Daniel Luft, Luka Schlegel, Kathrin Welker

UNIVERSITÄT TRIER

DFG

ALGORITHMIC OPTIMIZATION

www.alop.uni-trier.de

Overview

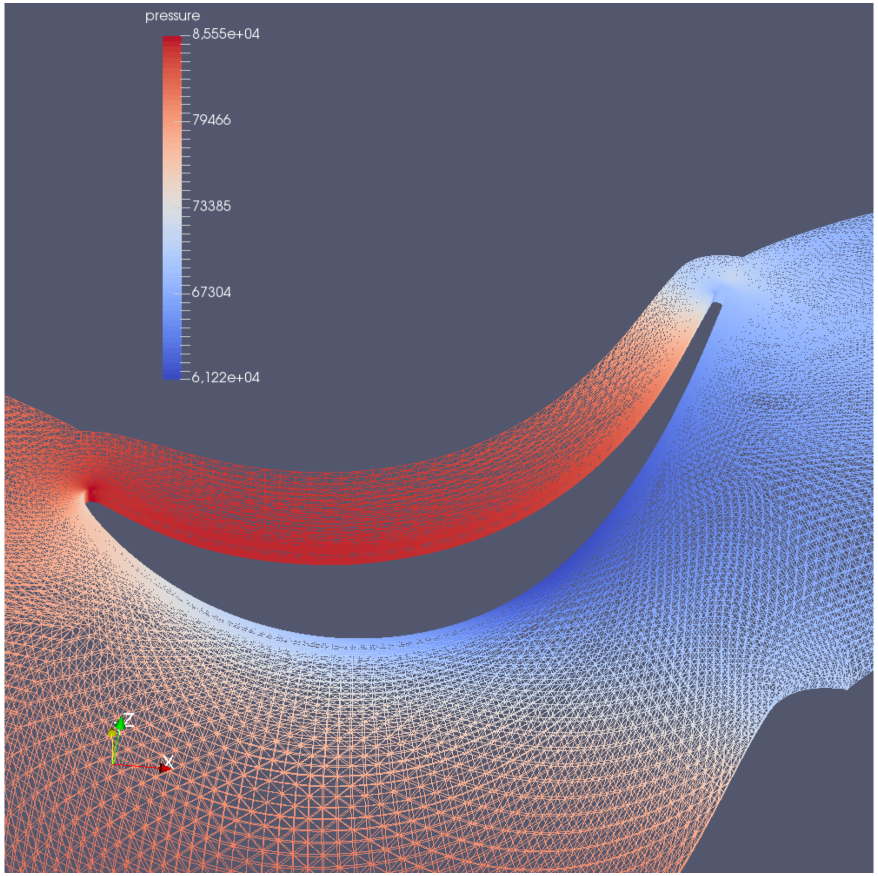
- Motivation from application projects
- Linear perspective to shape optimization
- Pre-shape calculus
- Further aspects

Turbomachinery design

Supported by BMBF: Uni Wuppertal, Siemens, DLR, Uni Trier (D. Luft)

Goal: find shapes minimizing low cycle fatigue

Model: RANS together with stochastic damage model



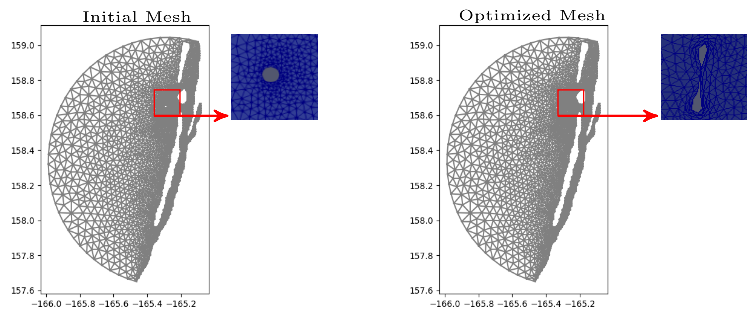
Pressure field for T106A

Shape Optimization for Mitigating Coastal Erosion

Supported by DFG: University Cheikh Anta Diop of Dakar (Diaraf Seck), Senegal, and Uni Trier (L. Schlegel)

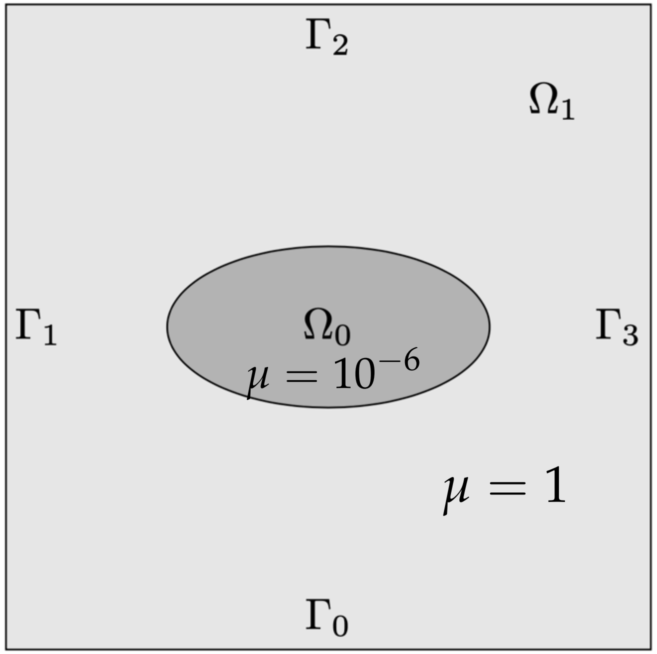
Goal: find shapes which mitigate coastal erosion

Model: shallow water combined with Exner's law



Langue de Barbarie: SWE results

Interface problem as test case



Dirichlet = 1

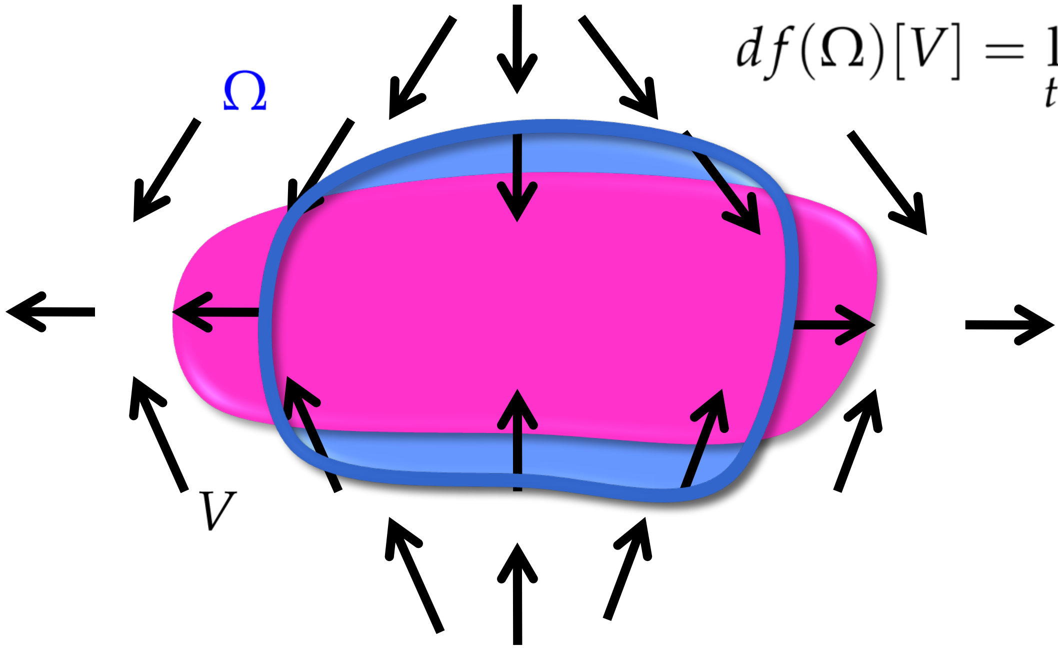
Dirichlet = 0

$$\min_{(u,\Omega)} \quad \frac{1}{2}\int_\Omega (u-z)^2 \, \mathrm{d}\,x + \frac{\alpha}{2} R(\Omega)$$

$$\int_\Omega \mu \, \langle \nabla u, \nabla v \rangle_2 - f v \;\; \mathrm{d}\,x = 0 \quad \forall v \in H^1_{(\Gamma_0 \cup \Gamma_2)}(\Omega)$$

Shape derivative for free node parametrizations

$$df(\Omega)[V] = \lim_{t \to 0} \frac{f(T_{t,V}(\Omega)) - f(\Omega)}{t}$$



$T_{t,V}(\Omega)$, where $T_{t,V}(x) = x + t \cdot V(x)$

Example: simple objective, no PDE constraint

$$f(\Omega_t) = \int_{\Omega_t} g(x)dx \qquad \Omega_t = T_{t,V}(\Omega_o)$$

$$T_{t,V}(x) = x + t \cdot V(x)$$

directional derivative in direction V:

$$df(\Omega_t)[V] := \left.\frac{d}{dt}\right|_{t=0} f(\Omega_t) = \left.\frac{d}{dt}\right|_{t=0} \int_{\Omega_t} g(x)dx$$

$$= \left.\frac{d}{dt}\right|_{t=0} \int_{\Omega_o} g(T_{t,V}(x))|det(DT_{t,V}(x))|dx$$

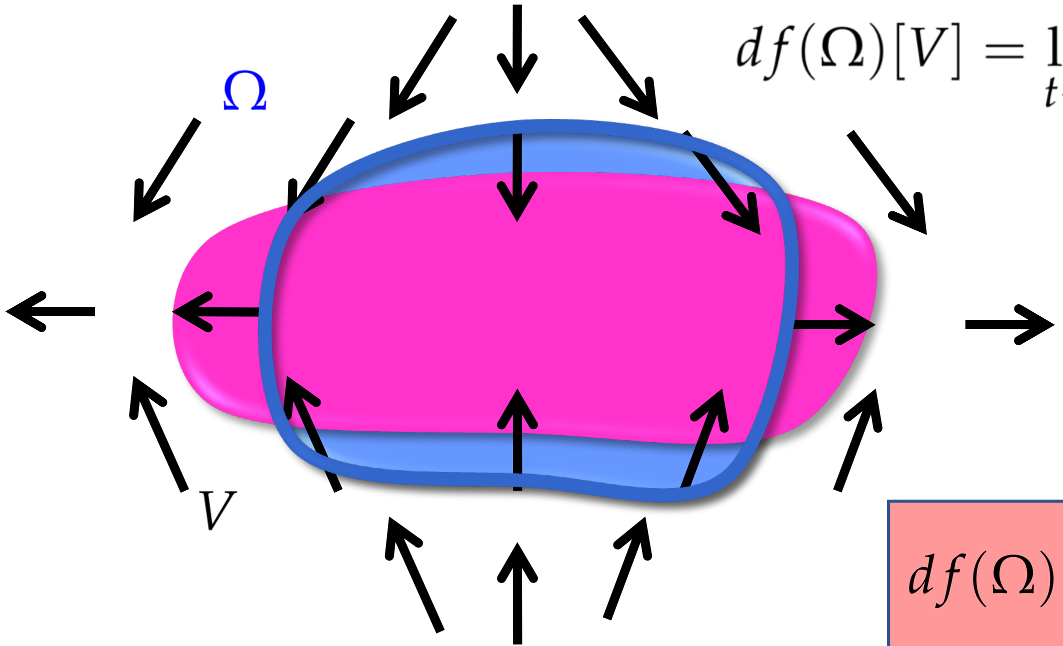
$$= \int_{\Omega_0} \frac{d}{dt}\Big|_{t=0} g(T_{t,V}(x))|det(DT_{t,V}(x))|dx$$

$$= \int_{\Omega_0} \nabla g(x)^\top V(x) + g(x) \cdot \underbrace{tr(DV(x))}_{divV(x)} dx$$

$$= \int_{\Omega_0} div(g(x)V(x))dx$$

$$= \int_{\partial\Omega_0} V(x)^\top \vec{n}(x) \cdot g(x)dx \quad \text{(Gauss)}$$

Shape derivative for free node parametrizations



$$df(\Omega)[V] = \lim_{t\to 0} \frac{f(T_{t,V}(\Omega)) - f(\Omega)}{t}$$

$$df(\Omega)[V] = \int_{\partial\Omega} g < V, n > ds$$

$T_{t,V}(\Omega)$, where $T_{t,V}(x) = x + t \cdot V(x)$

(Hadamard)

Shape derivative for interface problem with perimeter regularization

$$
\begin{aligned}
& dJ(\Omega)[V] \\
&= \int_{\Omega} \operatorname{div}(V) \left(\frac{1}{2}(u-z)^2 + \mu \nabla u^\top \nabla \lambda - \lambda f \right) - (u-z) \nabla z^\top V \\
&\qquad - \mu \nabla u^\top (DV + DV^\top) \nabla \lambda \, dx + \alpha \int_{\partial\Omega_0} \operatorname{div}_\Gamma V \, ds \\
&= \int_{\partial\Omega_0} \langle V, n \rangle \left([\![\mu]\!] \nabla u_1^\top \nabla \lambda_2 + \alpha \kappa \right) ds
\end{aligned}
$$

Some background

- Shape derivative -> descent direction, firstly shown to support convergence to solution in [Hintermüller, 2005]
- Lots of theory on characterization of shapes and optimality conditions by, e.g., Allaire, Delfour, Zolesio, Sokolowski, Trouve, Mäkinen, Eppler, Harbrecht, Schneider, Sturm, ...
- Only very view investigations on algorithmic aspects besides finite dimensional cases.

Standard Optimization Algorithm

[Schulz/Siebenborn/Welker, SIOP 2016]

- Solve $a(U,V) = dJ(\Omega^k)[V]\,,\ \forall V$, where *a(.,.)* from elasticity, resulting in a Steklov-Poincarè type metric on the shape boundary
- Improve *U* in a l-BFGS double loop (based on previous steps)
- Update shape together with whole mesh on domain

$$\Omega^{k+1} = (id + \tau \cdot U)(\Omega^k)$$

- Steklov-Poincarè type metric in line with Hessian analysis

Elasticity with varying Lame

$$a(U,V) := \int_{\Omega} \sigma(U) : \epsilon(V) dx = dJ[V],\ \forall V \in H^1$$

$$\sigma(Z) := \lambda \mathrm{tr}(\epsilon(Z)) I + 2\mu\epsilon(Z)$$

$$\epsilon(Z) := \frac{1}{2}(\nabla Z + \nabla Z^\top)$$

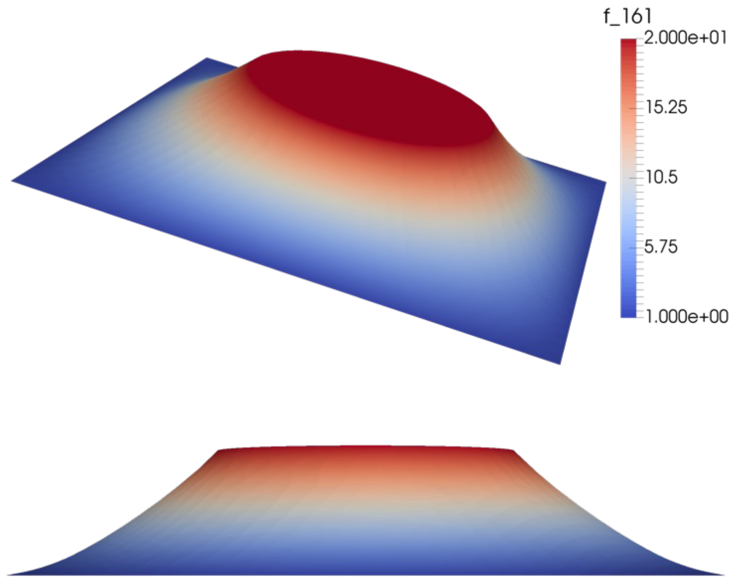
$$\lambda = 0$$

μ solves the elliptic BVP:

$$\triangle \mu = 0\ ,\ \text{in } \Omega_1$$

$$\mu = \mu_{\max}\ ,\ \text{on } \Gamma_{int}$$

$$\mu = \mu_{\min}\ ,\ \text{on } \partial\Omega$$



Shape Hessian for Newton performance

$$d^2J(\Omega_2)[V,W] = \frac{d}{ds}\Big|_{s=0+}\frac{d}{dt}\Big|_{t=0+} J((id+sV)\circ(id+tW)(\Omega_2)) : W^{1,\infty}(\Omega)\times W^{1,\infty}(\Omega)\to\mathbb{R}$$

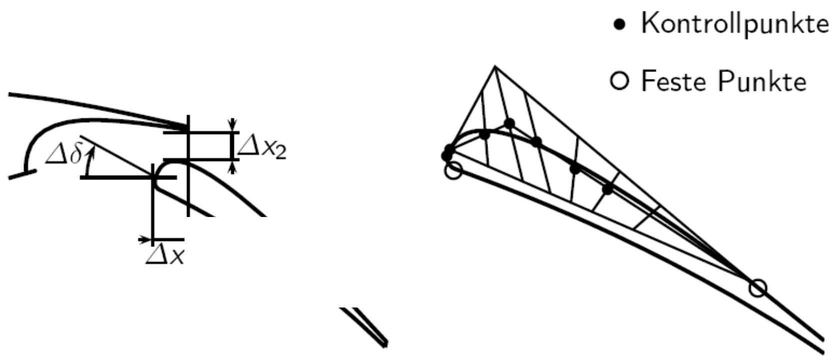
where $\quad (id+sV)\circ(id+tW) = id+tV+sW+stV\circ W$

- Bilinear mapping - obviously not symmetric (outside solution)
- Symmetric Riemannian variant (Schulz 2014)
- Related problem [Eppler/Harbrecht/Schneider, SICON 2007]:

$$d^2J(\Omega_2)[\alpha n|_\Gamma, \alpha n|_\Gamma] \geq c\|\alpha\|^2_{H^{1/2}(\Gamma)}$$

- Dirichlet-to-Neumann map for 2nd order may give appropriate scalar product [Schulz/Siebenborn/Welker, SIOP 2016]

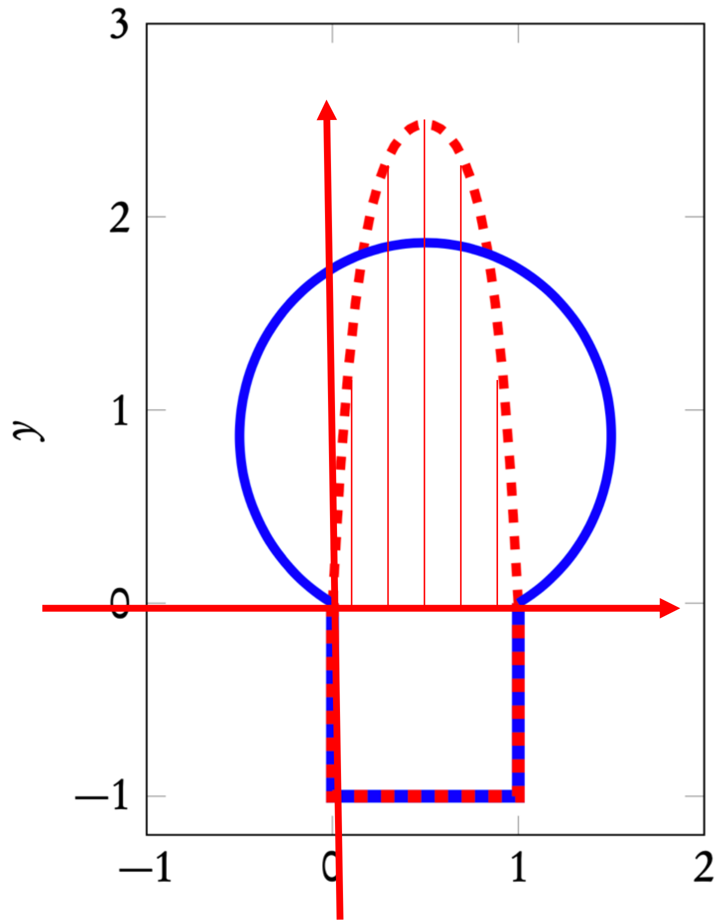
Shape concepts: linear



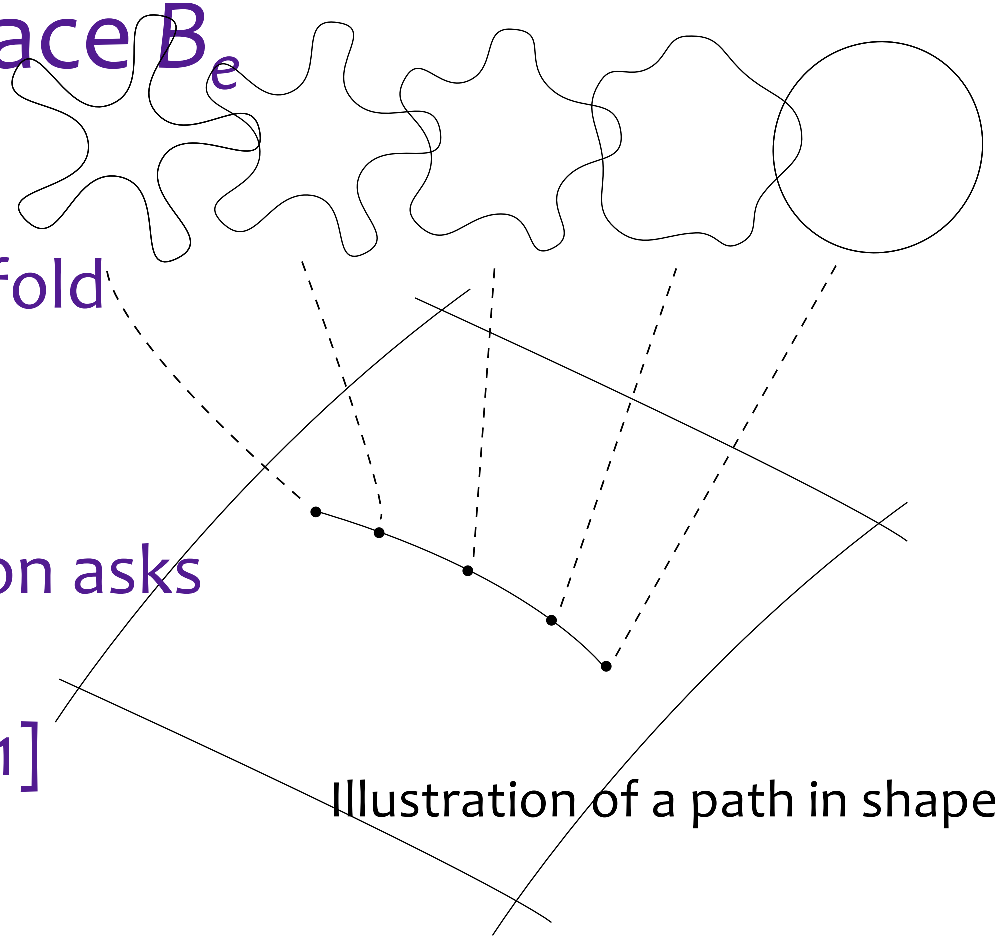
- Finite shape parametrization in many industrial shape optimizations
  - Pro: vector space setting, fits in CAD framework
  - Con: complexity inevitably increases with number of parameters, mesh sensitivities can become expensive, set of reachable shapes is restricted

Method of mapping

- Essentially considers the deformed boundary as a graph
- Limits reachability
- Example: Dido Problem: method of mapping vs. solution



Nonlinear Shape space B_e



- Smooth shapes with manifold structure [Schulz, 2014]

- Algorithmic implementation asks for less smoothness

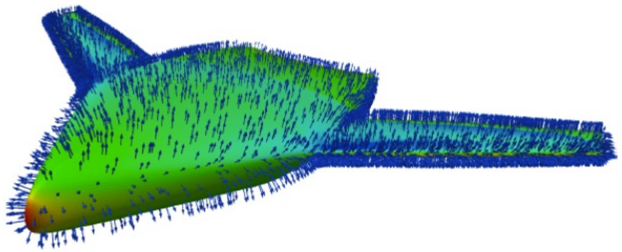
-> Diffeologies [Welker, 2021]

Illustration of a path in shape space

Algorithmic implementation of shape calculus

Non-CAD approach built on shape calculus

- avoids cons of parametric approach, can be very efficient
- Essentially, deformation vector field is added to current mesh



→ *Idea: let‘s use the deformation vector field itself as optimization variable!*

Shape algo with focus on deformations

Algorithm Steepest Descent Shape Deformation Optimization

$k := 0$, initialize $T^0 = I, \varepsilon$

repeat

determine V^k solving $b_k(V^k, Z) = -dJ(\Omega^k)[Z]\,,\ \forall Z \in \mathcal{H}$

Line search: find $t^k \approx \arg\min_t J((I + tV^k) \circ T^k(\Omega^0))$

$T^{k+1} := (I + t^k V^k) \circ T^k$

$k := k + 1$

until $\|V^k\| \leq \varepsilon$

Linear View on Shape Optimization

[S. Schmidt and V. Schulz, arXiv: 2203.07175, 2023: acc. with SICON]

- Consider all domains of deformations of an initial domain

$$\Omega = T(\Omega^0), \quad T \in \mathcal{H} \text{ deformation}$$

- Define objective on deformation Hilbert space $\mathcal{H}$:

$$f(T) := J(T(\Omega^0))$$

- Derivative related to shape derivative

$$dJ(\Omega)[Z] = \frac{d}{dt}\Big|_{t=0+} J((I+tZ)(\Omega)) = \frac{d}{dt}\Big|_{t=0+} f((I+tZ)\circ T) = f'(T)[Z\circ T]$$

Consequences

- Shape derivative is equivalent to deformation derivative – in local coordinates

$$\nabla f(T) = \nabla J(T(\Omega^0)) \circ T$$

- Taylor series in linear space with linear shape Hessian

$$J((I+V)(\Omega)) = J(\Omega) + b(\nabla J(\Omega), V)_T + \frac{1}{2} J''(\Omega)[V,V] + \mathcal{O}(\|V\|^3)$$

$$J''(\Omega)[V,W] := \frac{d}{ds_1}\Big|_{s_1=0+} \frac{d}{ds_2}\Big|_{s_2=0+} J((I + s_1 V + s_2 W)(\Omega)).$$

$$d^2 J(\Omega)[V,W] = \frac{d}{ds_1}\Big|_{s_1=0+} \frac{d}{ds_2}\Big|_{s_2=0+} J((I + s_1 V) \circ (I + s_2 W)(\Omega))$$

Central relation

$$J''(\Omega)[V,W] = d^2J(\Omega)[V,W] - dJ(\Omega)[(DV)W]$$

- Linear second shape derivative enables transfer of NLP algorithms and their convergence properties to shape optimization.
- Standard shape Hessian is non-symmetric perturbation of second shape derivative, vanishing at the solution.

-> Derive shape Hessian in the standard process and forget all terms containing *(DV)W* on the way.

Challenge: Hessian with huge nullspace!

- Hessian defined by $g(H(T)V,W) := f''(T)[V,W]\,,\ \forall V,W \in \mathcal{H}$

- Newton with pseudo-inverse

$$T^{k+1} = T^k + V^k \circ T^k\,,\ \text{where } V^k = -H(T^k)^+ \nabla f(T^k)$$

- Locally quadratic convergence of residual [Deuflhard, 2004]
- MP pseudo-inverse depends on choice of scalar product [Groetsch 1977]

[S. Schmidt and V. Schulz, arXiv: 2203.07175]

THEOREM *Let $(\mathcal{H}, g)$ be a Hilbert space with inner product g. We assume that the linear operator H defined on $\mathcal{H}$ has closed range and is not necessarily invertible. When solving the equation $HV = b$ with $b \in \mathcal{R}(H)$, we obtain $\hat{V} := H^{+}b$ as the minimum norm solution, where H^{+} is the Moore-Penrose pseudoinverse operator. Then, the vector $\hat{V}$ is also the unique solution of the optimization problem*

$$\min_{V} \; g(V,V)$$
$$s.t. \quad HV = b$$

Furthermore, if H is self-adjoint in the scalar product g and positive semidefinite, then the vector $\hat{V}$ can be computed as the limit of the solutions V_ε of the following family of linear-quadratic problems parameterized by $\varepsilon > 0$. For

$$V_\varepsilon := \arg\min_{V} \frac{1}{2} g(HV, V) - g(b, V) + \frac{\varepsilon}{2} g(V, V)$$

there holds $\lim_{\varepsilon \to 0} V_\varepsilon = \hat{V}$.

(Alternative: Krylov subspace methods)

Weak view on PDE constrained shape calculus

$$\min J(u,\Omega)$$
$$\text{s.t. } a(u,\lambda;\Omega)+f(\lambda;\Omega)=0\,,\ \forall\lambda\in\Lambda$$

Lagrangian:

$$\mathscr{L}(u,\lambda;\Omega)=J(u,\Omega)+a(u,\lambda;\Omega)+f(\lambda;\Omega)$$

Weak necessary conditions:

$$d\mathscr{L}(u,\lambda;\Omega)[d_m u[V],d_m\lambda[V],V]=0$$
$$\forall(d_m u[V],d_m\lambda[V],V)\in U\times\Lambda\times H^1(D,D)$$

-> Shape derivative of Lagrangian leads to material derivatives to be used as test functions!

Alternative: „averaged adjoints“ due to K. Sturm

Newton method in weak formulation

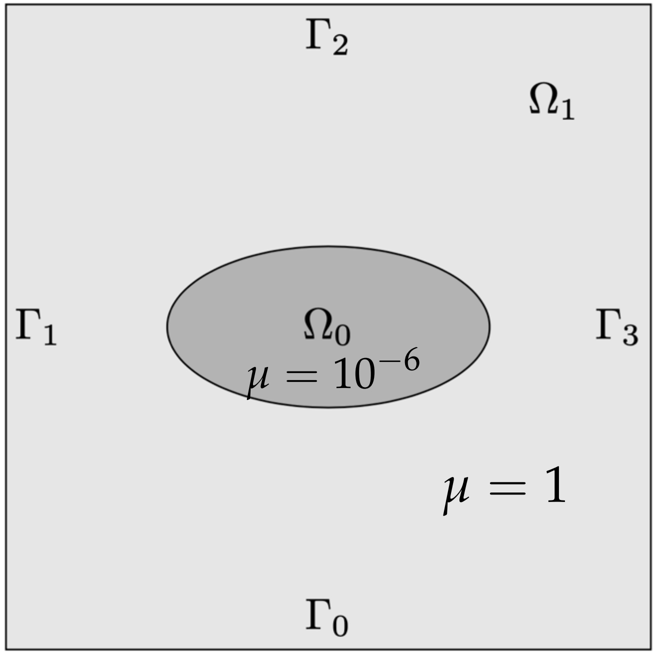
- Regularized KKT system

$$\mathscr{L}''(u_k,\Omega_k,\lambda_k)[\tilde{u},\tilde{V},\tilde{\lambda}][\hat{u},\hat{V},\hat{\lambda}] + \frac{\varepsilon}{2} b_{\Omega_k}(\tilde{V},\hat{V}) = -\,\mathrm{d}\,\mathscr{L}(u_k,\Omega_k,\lambda_k)[\tilde{u},\tilde{V},\tilde{\lambda}]$$
$$\forall(\tilde{u},\tilde{q},\tilde{\lambda}) \in \mathrm{CG}_{r_1} \times \mathrm{CG}^2_{r_2} \times \mathrm{CG}_{r_1}$$

- With scalar product

$$b_{\Omega_k}(W,V) = \int_{\Omega} \left(\langle W,V\rangle_F + \frac{1}{2}\langle \nabla W, \nabla V\rangle_F \right) \mathrm{d}\,x$$

Interface problem as test case



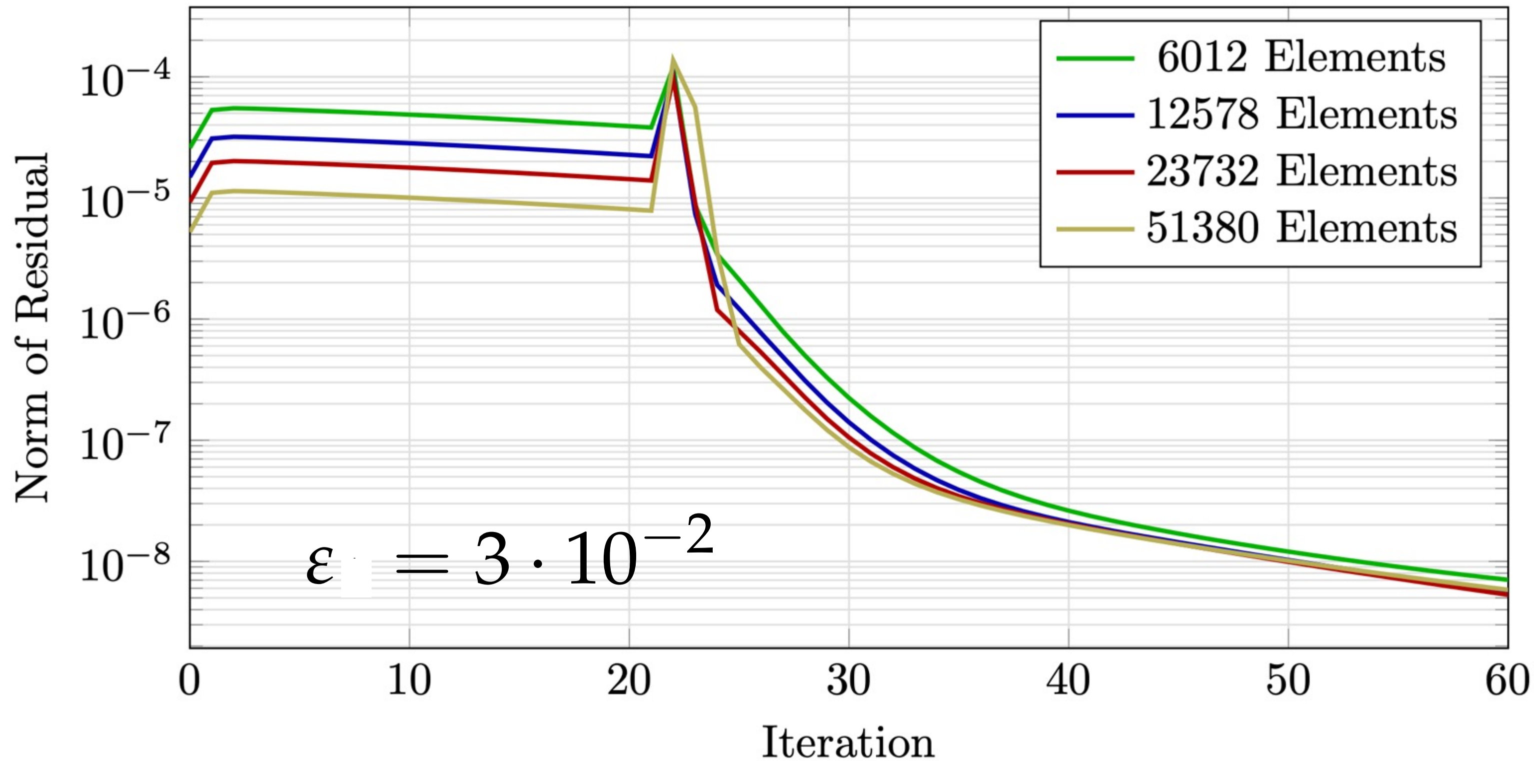
Dirichlet = 1

Dirichlet = 0

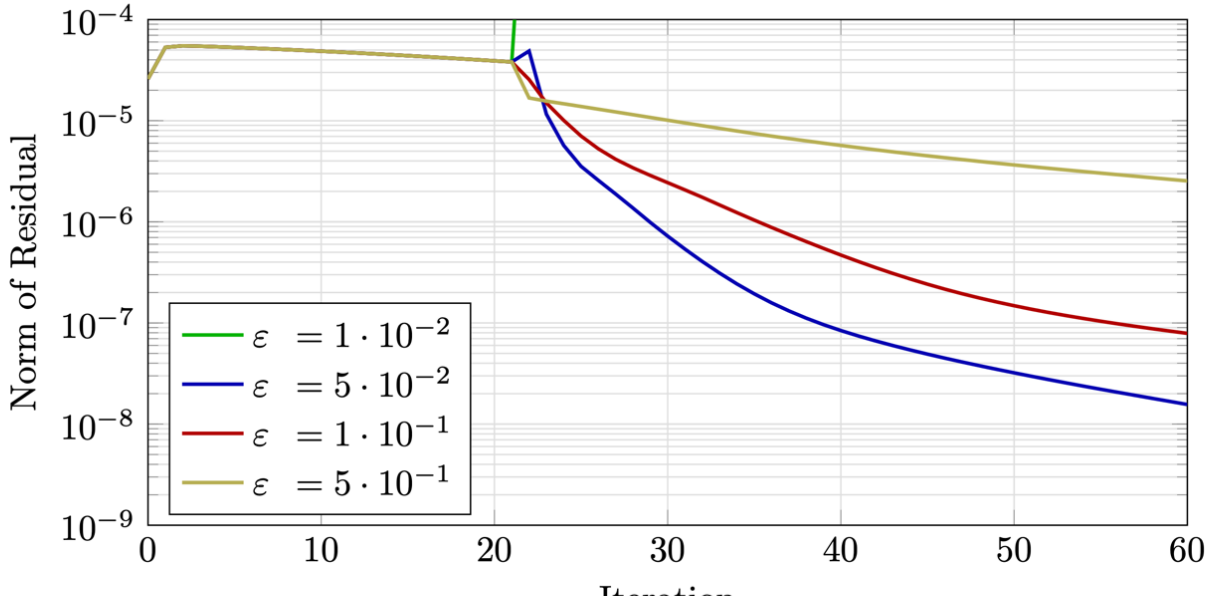
$$\min_{(u,\Omega)} \quad \frac{1}{2}\int_\Omega (u-z)^2 \, \mathrm{d}\, x + \frac{\alpha}{2} R(\Omega)$$

$$\int_\Omega \mu \, \langle \nabla u, \nabla v \rangle_2 - f v \;\; \mathrm{d}\, x = 0 \quad \forall v \in H^1_{(\Gamma_0 \cup \Gamma_2)}(\Omega)$$

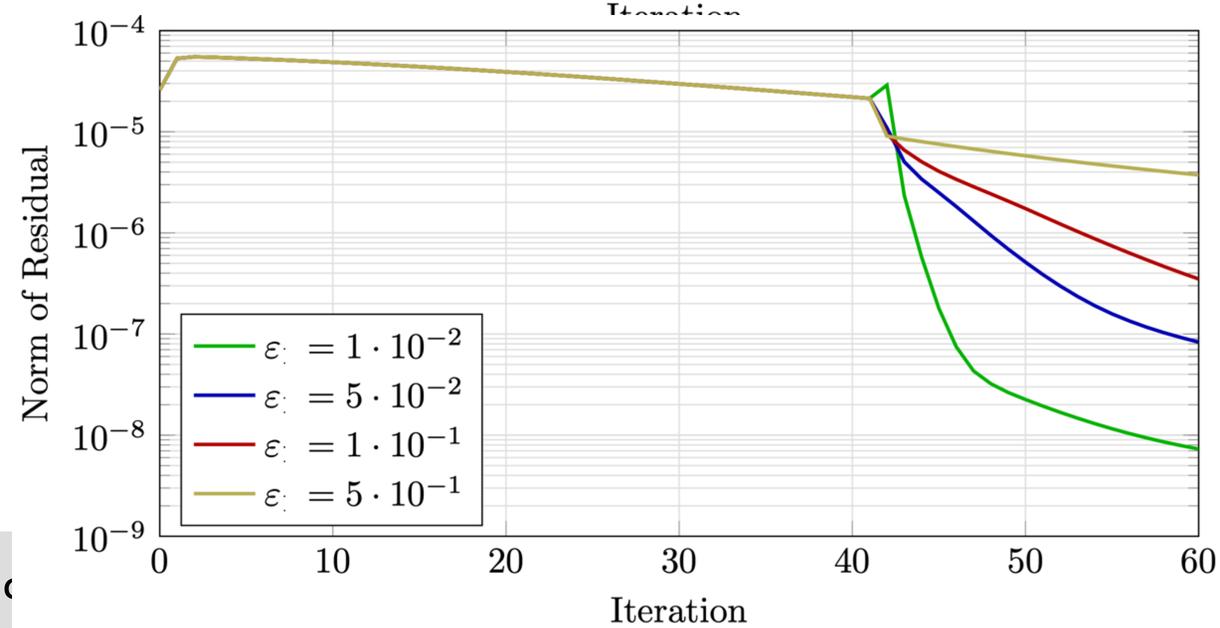
Numerical tests



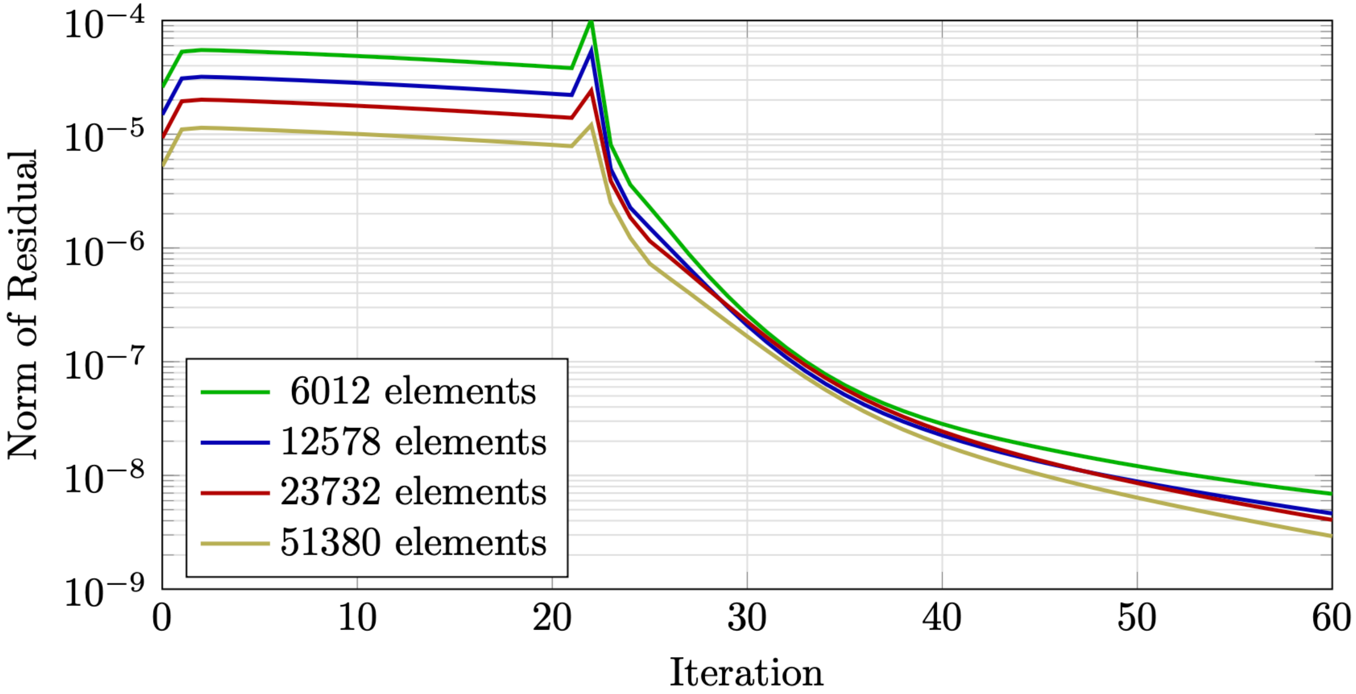
Effect of ε



For ambitious regularization, one must start closer



Adaptive choice of ε



Test for adaptivity based on $\varepsilon_k = \mathcal{O}(\|V^k\|)$. *The proportionality factors are* 800, 1600, 3200 *and* 6400 *as the mesh resolution increases.*

Pre-Shape Calculus

Daniel Luft and Volker Schulz

arXiv:2012.09124, arXiv:2103.15109

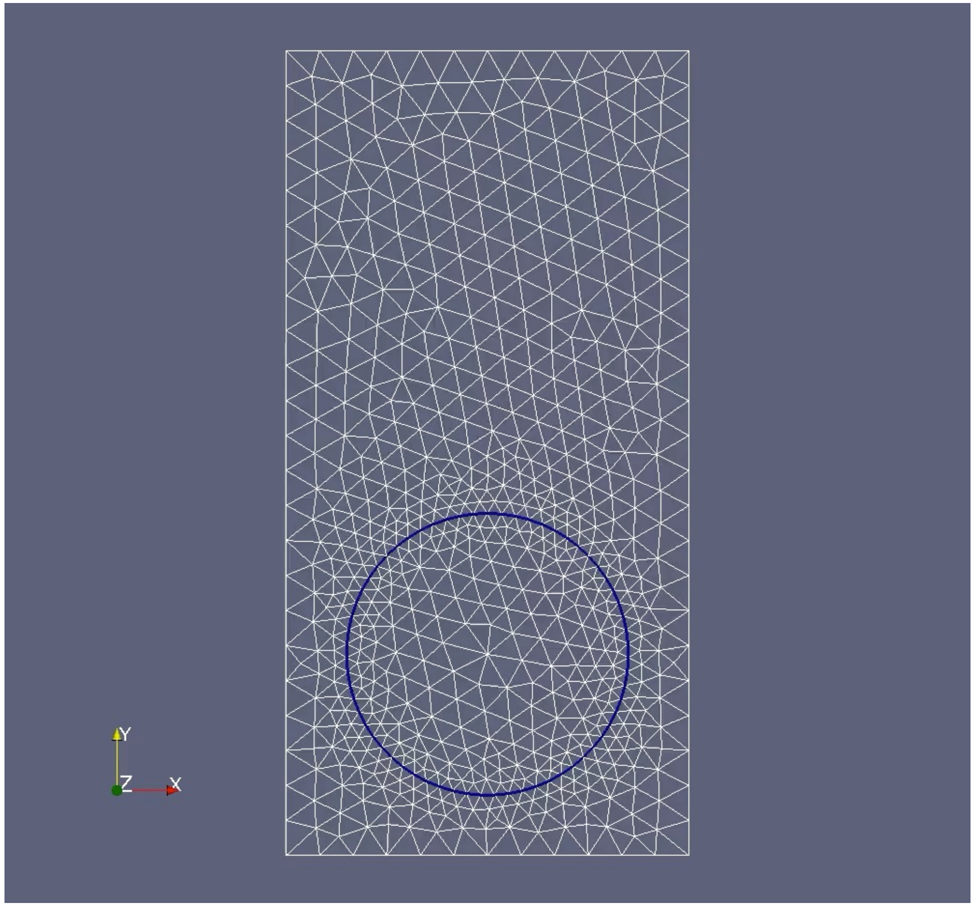
Published in Journal Control & Cybernetics 2021/22

Problem with normal deformations

- Hadamard: tangential movements are irrelevant for shape derivatives!

- Large deformations: bad tangential resolution

-> improve this in an integrated way



Shape spaces and functionals

[P.W. Michor, D. Mumford, 2006]

Let M be a n-dim. orient. path-conn. and compact C^∞-submanifold of $\mathbb{R}^{n+1}$. Then

$$B_e^n := \mathrm{Emb}(M, \mathbb{R}^{n+1}) / \mathrm{Diff}(M)$$

is called *space of smooth shapes.* Further

$$\mathcal{J} : B_e^n \to \mathbb{R}$$

is called *shape functional.*

Shape space B_e

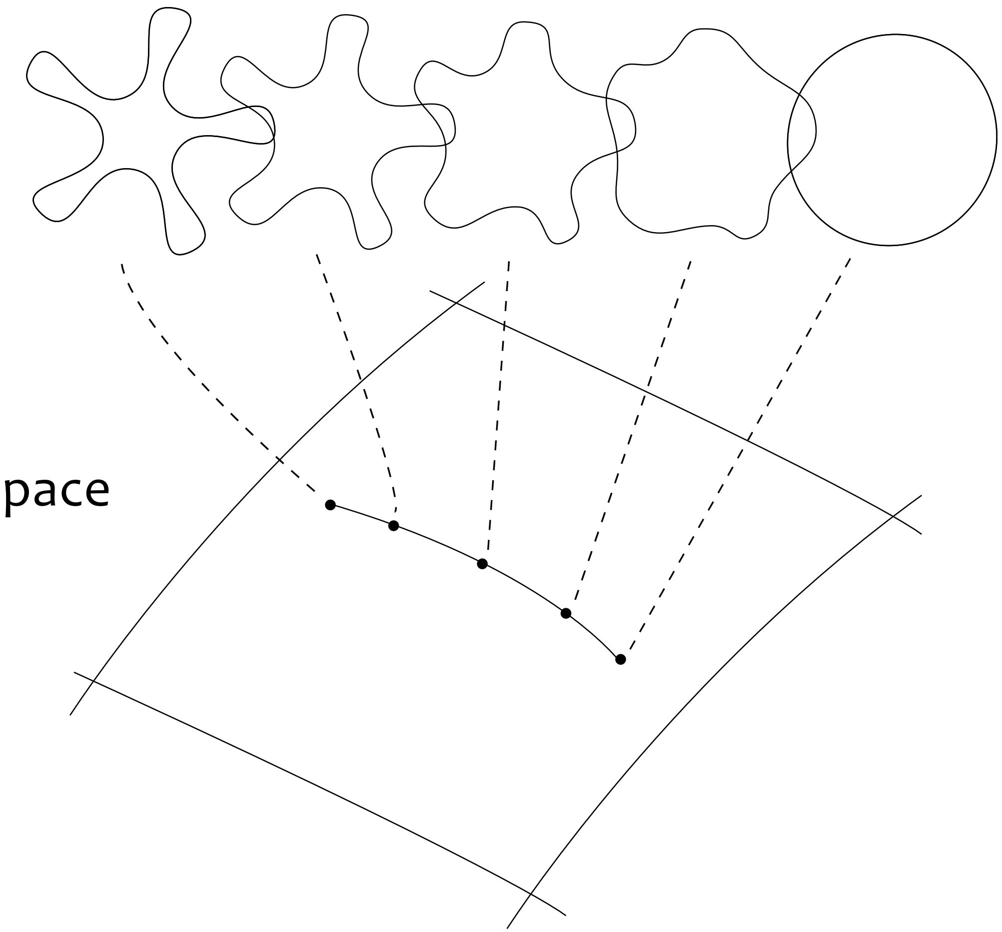


Illustration of a path in shape space

Shape calculus

Let $\mathcal{J}$ be a shape functional, $\Gamma \in B_e^n$ and $V \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$. Then the family of shapes

$$\Gamma_t(x) := \{x + t \cdot V(x) \,|\, x \in \Gamma\}$$

is called *perturbation of identity* of Γ *in direction* V. The limit

$$\mathcal{D}\mathcal{J}(\Gamma)[V] := \lim_{t \to 0+} \frac{\mathcal{J}(\Gamma_t) - \mathcal{J}(\Gamma)}{t}$$

is called shape derivative for $\mathcal{J}$ in Γ in direction V, if it exists and is linear and bounded in V.

Hadamard: $\mathcal{D}\mathcal{J}(\Gamma)[V] = \int_\Gamma g \cdot (V, \vec{n}) ds$

Pre-Shape spaces and functionals

[M. Bauer, M. Bruveris, and P.W. Michor, 2014]

Let M be a n-dim. orient. path-conn. and compact C^∞-submanifold of $\mathbb{R}^{n+1}$. Then

$$\mathrm{Emb}(M, \mathbb{R}^{n+1})$$

is called *pre-shape space.* Further

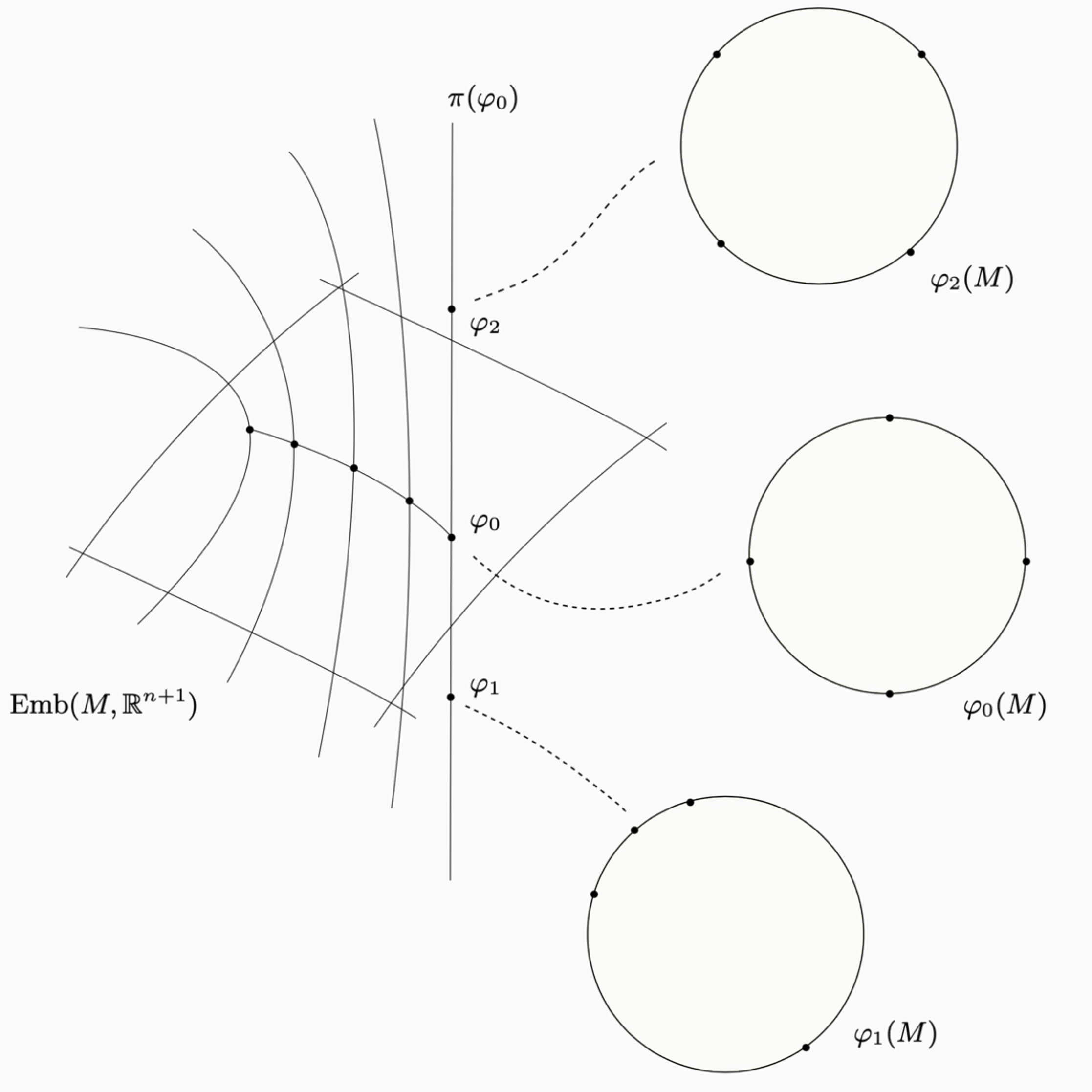
$$\mathfrak{J} : \mathrm{Emb}(M, \mathbb{R}^{n+1}) \to \mathbb{R}$$

is called *pre-shape functional.*

Pre-Shape Space

$\mathrm{Emb}(M, \mathbb{R}^{n+1})$

Illustration of a path in pre-shape space



Pre-shape calculus

Let $\mathfrak{J}$ be a pre-shape functional, $\varphi \in \mathrm{Emb}(M, \mathbb{R}^{n+1})$, $V \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$
Then the family of functions

$$\varphi_t := \varphi + t \cdot V \circ \varphi$$

is called (pre-shape) *perturbation of identity* of φ *in direction* V. The limit

$$\mathfrak{D}J(\Gamma)[V] := \lim_{t \to 0+} \frac{\mathfrak{J}(\varphi_t) - \mathfrak{J}(\varphi)}{t}$$

is called pre-shape derivative for $\mathfrak{J}$ in φ in direction V, if it exists and is linear and bounded in V.

Relations

- Every shape differentiable function is also pre-shape differentiable via canonical extension

$$\mathfrak{J} : \mathrm{Emb}(M, \mathbb{R}^{n+1}) \to \mathbb{R}\,,\ \varphi \mapsto \mathcal{J}(\pi(\varphi))$$

- Pre-shape material derivative $\mathfrak{D}_m$ is analogously defined

- Pre-shape calculus gives similar formulas as shape calculus

- There is also a pre-shape Hessian

Pre-shape structure theorem

arXiv:2012.09124

Let $\mathfrak{J} : \mathrm{Emb}(M, \mathbb{R}^{n+1}) \to \mathbb{R}$ be a pre-shape differentiable pre-shape functional and let $\varphi \in \mathrm{Emb}(M, \mathbb{R}^{n+1})$

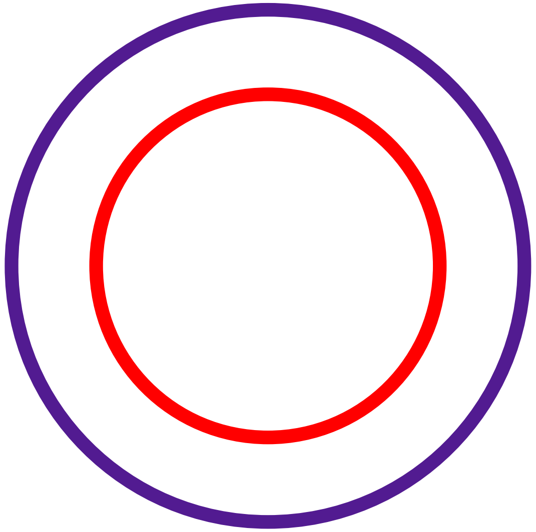
Then there exist functions $g^n : \varphi(M) \to \mathbb{R}, g^t : C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \to \mathbb{R}$ depending on φ such that

$$\mathfrak{D}J(\varphi)[V] = \int_{\varphi(M)} g^n \cdot (V, \vec{n}) + g^t(V) ds, \qquad \mathrm{supp}(g^t|_{\varphi(M)}) \subset T_{\varphi(M)}$$

Example S¹

$$\min_{\varphi \in \mathrm{Emb}(S^1,\mathbb{R}^2)} \frac{1}{2}\int_{S^1} |\varphi - \tilde{\varphi}|^2 \, \mathrm{d}s \;=: \mathfrak{J}(\varphi)$$

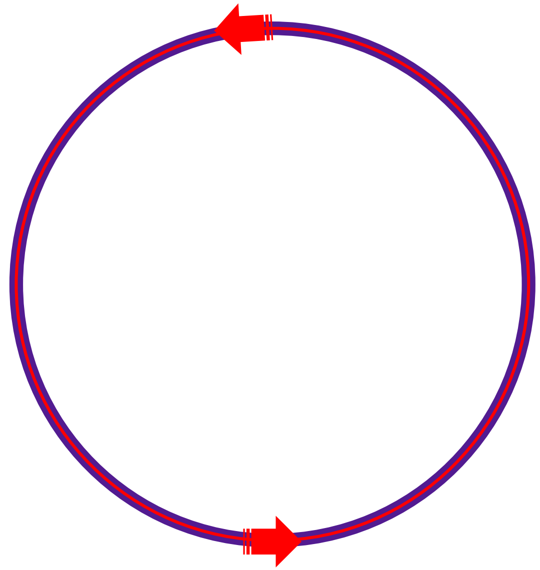
$$\begin{aligned}\mathfrak{D}\mathfrak{J}(\varphi)[V] &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{1}{2}\int_{S^1} |\varphi_t - \tilde{\varphi}|^2 \, \mathrm{d}s \\ &= \frac{1}{2}\int_{S^1} \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \langle \varphi + t\cdot V\circ\varphi - \tilde{\varphi}, \varphi + t\cdot V\circ\varphi - \tilde{\varphi}\rangle \, \mathrm{d}s \\ &= \int_{S^1} \langle \varphi - \tilde{\varphi}, V\circ\varphi\rangle \, \mathrm{d}s.\end{aligned}$$



example

$$\tilde{\varphi} = \alpha \cdot \mathrm{id} \;\Rightarrow\; \mathfrak{D}J(\mathrm{id})[V] = \int_{S^1} (1-\alpha)\cdot\langle V, \vec{n}\rangle ds$$

Rotation of S¹



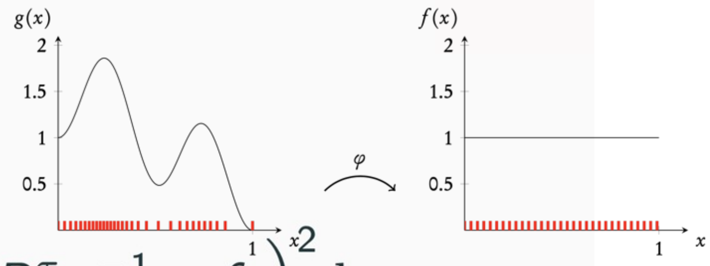
$$\tilde{\varphi} : S^1 \to \mathbb{R}^2, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathfrak{D}\mathfrak{J}(\varphi_{id})[V] = \int_{S^1} (1 - \cos(\alpha)) \cdot \langle n, V \rangle \, \mathrm{d}s + \int_{S^1} \sin(\alpha) \cdot \langle \tau, V \rangle \, \mathrm{d}s$$

Tracking

Let $g^M : M \to (0,\infty)$, $f_\varphi : \varphi(M) \to (0,\infty)$ be C^∞-functions. Further let

$$\int_{\varphi(M)} f_\varphi \, \mathrm{d}s = \int_M g^M \, \mathrm{d}s \quad \forall \varphi \in \mathrm{Emb}(M, \mathbb{R}^{n+1}).$$



Then the following problem

$$\min_{\varphi \in \mathrm{Emb}(M,\mathbb{R}^{n+1})} \frac{1}{2} \int_{\varphi(M)} \left(g^M \circ \varphi^{-1} \cdot \det D^\tau \varphi^{-1} - f_\varphi \right)^2 \mathrm{d}s$$

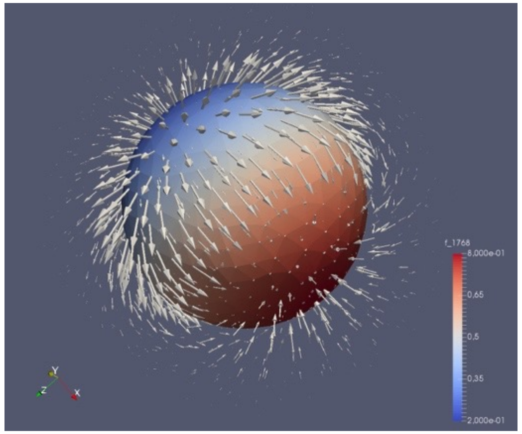
$$=: \mathfrak{J}^\tau(\varphi)$$

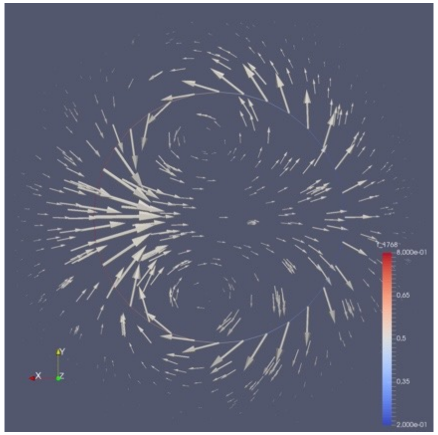
is called *Shape Parametrization Tracking Problem.*

Pre-shape derivative of tracking

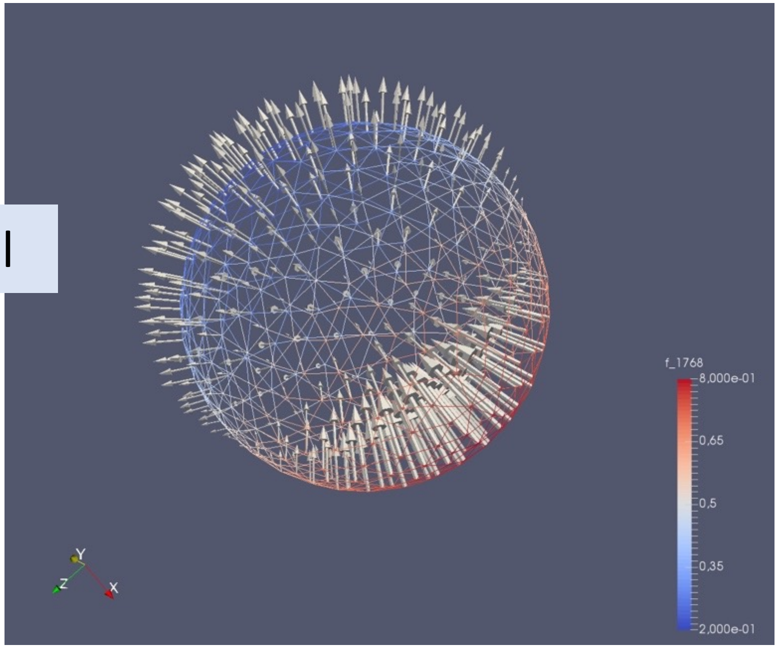
$$
\begin{aligned}
\mathfrak{D}\mathfrak{J}^\tau(\varphi)[V] = &-\int_{\varphi(M)} \frac{\dim(M)}{2} \cdot \Big(\big(g^M \circ \varphi^{-1} \cdot \frac{1}{\det D^\tau \varphi} \circ \varphi^{-1}\big)^2 - f_\varphi^2 \Big) \cdot \kappa \cdot \langle V, n \rangle \\
&\qquad + \Big(g^M \circ \varphi^{-1} \cdot \frac{1}{\det D^\tau \varphi} \circ \varphi^{-1} - f_\varphi \Big) \cdot \Big(\frac{\partial f_\varphi}{\partial n} \cdot \langle V, n \rangle + \mathcal{D}(f_\varphi)[V] \Big) \,\mathrm{d}s \\
&-\int_{\varphi(M)} \frac{1}{2} \cdot \Big(\big(g^M \circ \varphi^{-1} \cdot \frac{1}{\det D^\tau \varphi} \circ \varphi^{-1}\big)^2 - f_\varphi^2 \Big) \cdot \mathrm{div}_\Gamma(V - \langle V, n \rangle \cdot n) \\
&\qquad + \Big(g^M \circ \varphi^{-1} \cdot \frac{1}{\det D^\tau \varphi} \circ \varphi^{-1} - f_\varphi \Big) \cdot \nabla_\Gamma f_\varphi^T V \,\mathrm{d}s,
\end{aligned}
$$

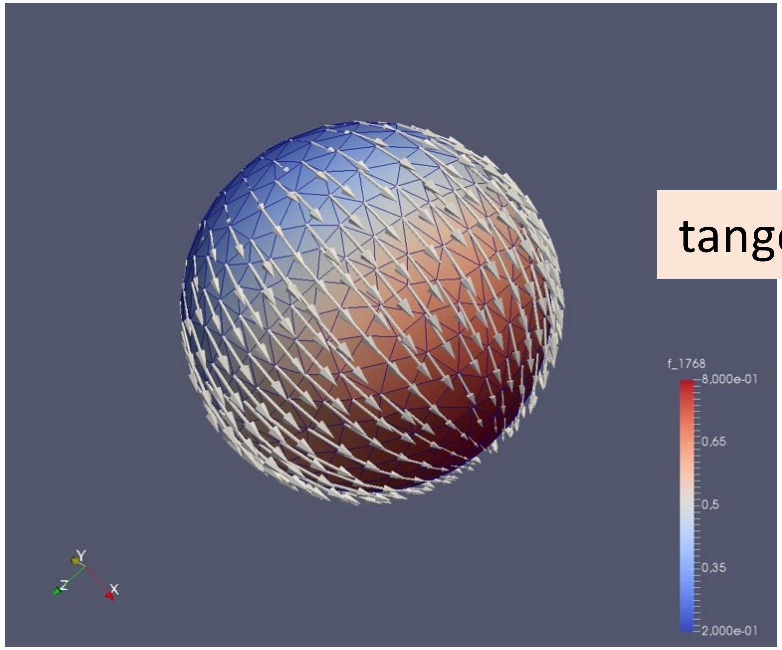
Illustration of the gradient components on S^2





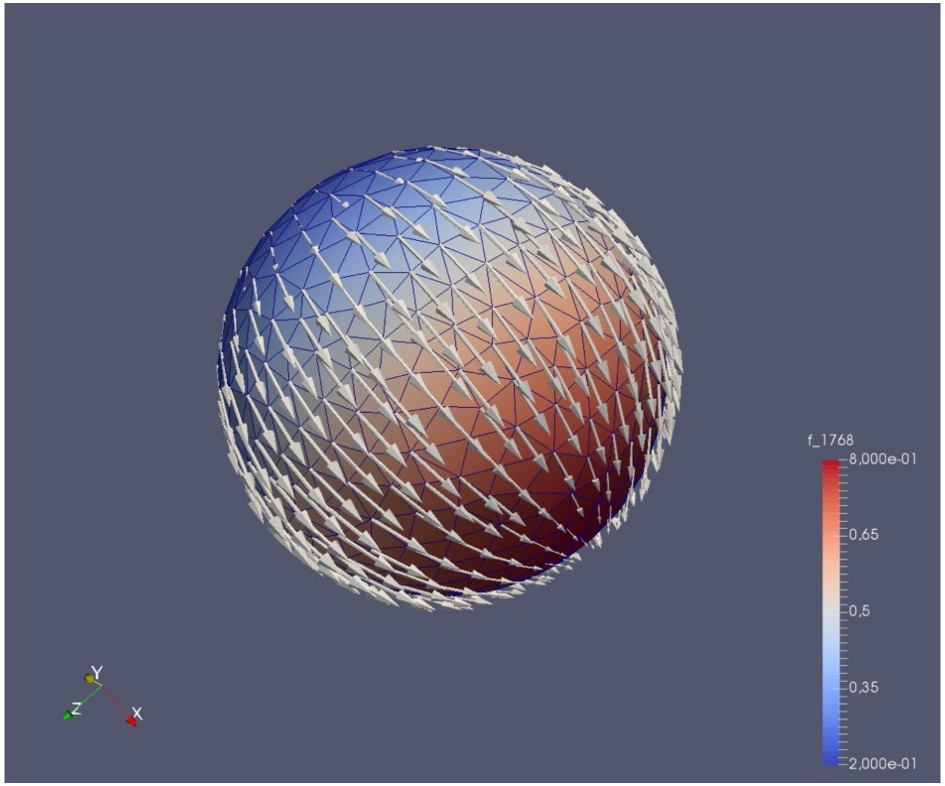
normal

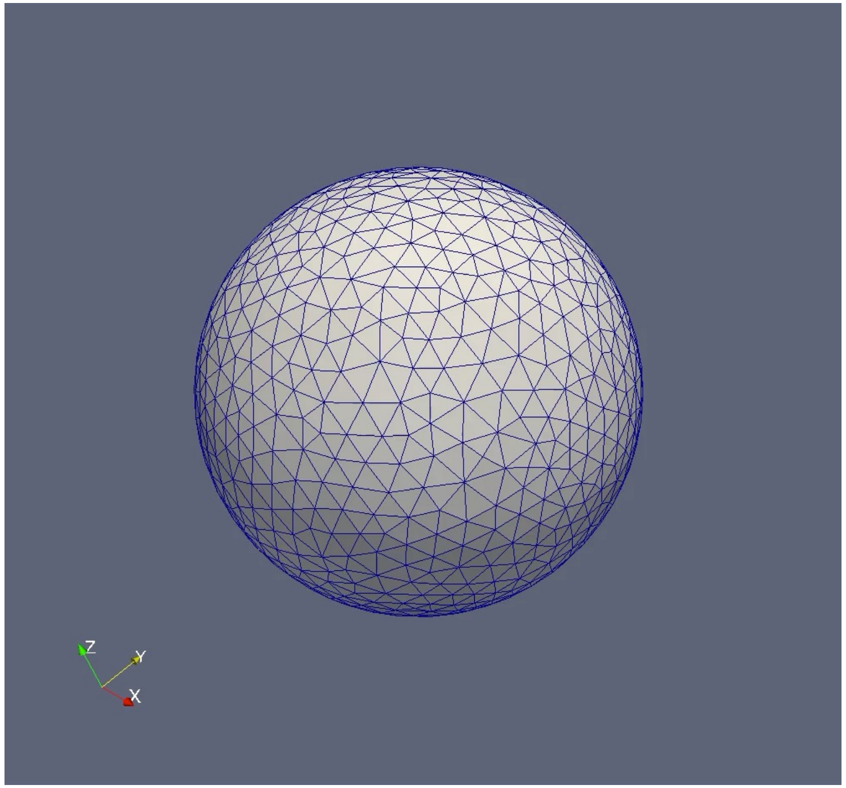




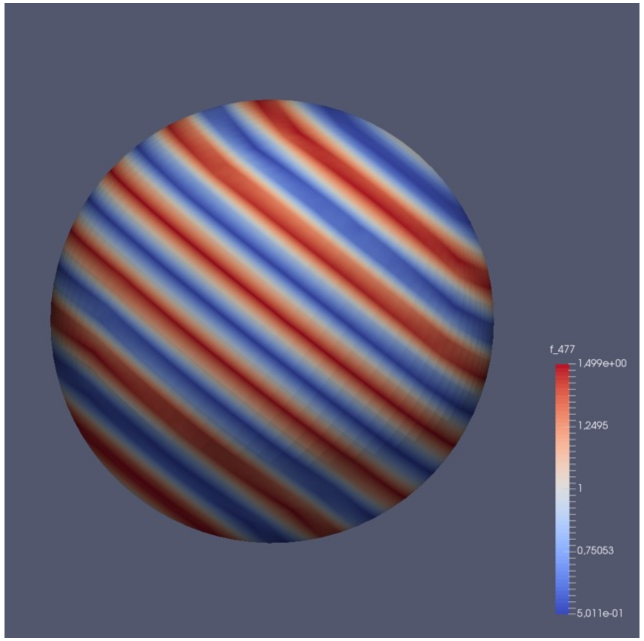
tangential

Deformation based on tangential component

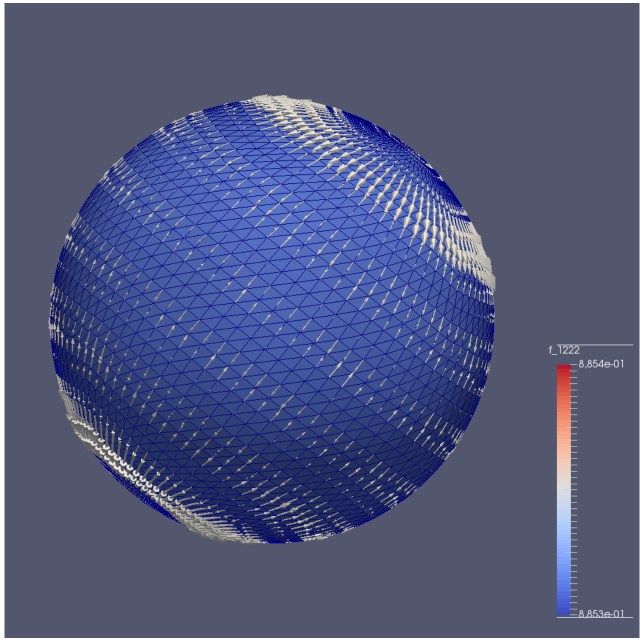




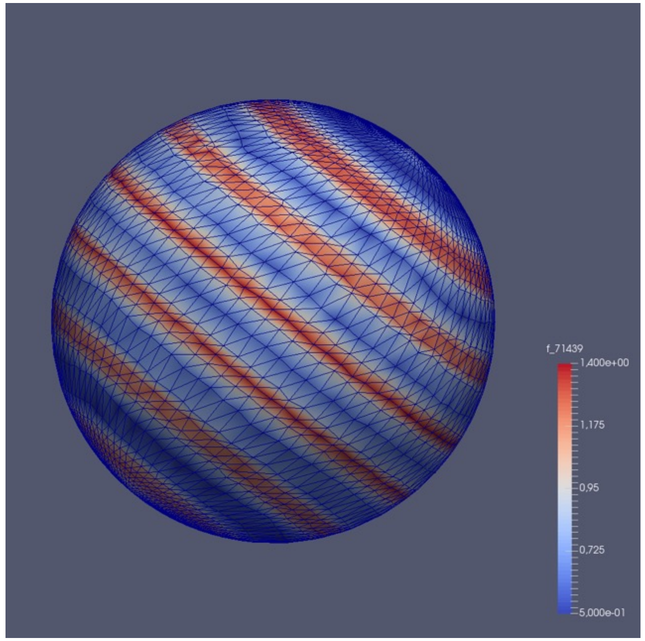
Yet another target distribution



target



- gradient



result

Bilevel shape and mesh optimization

$$\min_{\varphi \in \mathrm{Emb}(M, \mathbb{R}^{n+1})} \mathfrak{J}^{\tau}(\varphi)$$ <- tracking

$$s.t. \;\; \pi(\varphi) = \operatorname*{arg\,min}_{\Gamma \in B_e^n} \mathcal{J}(\Gamma).$$ <- shape

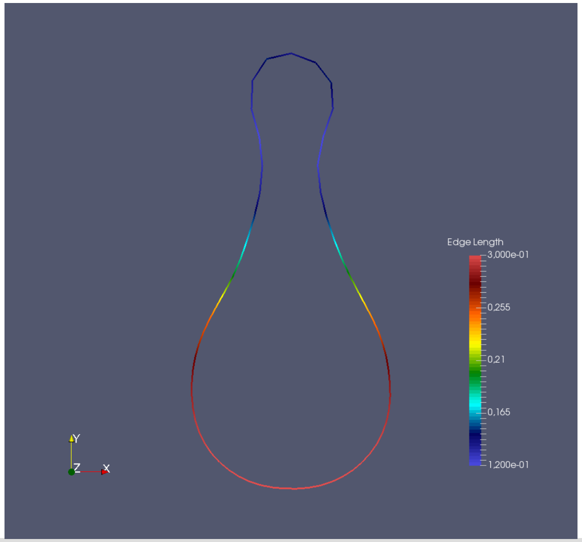
-> Generalization to Volume mesh analogously

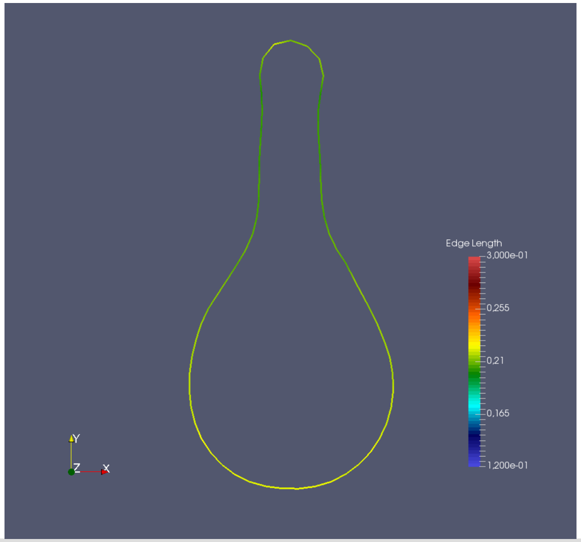
Consistent one level implementation

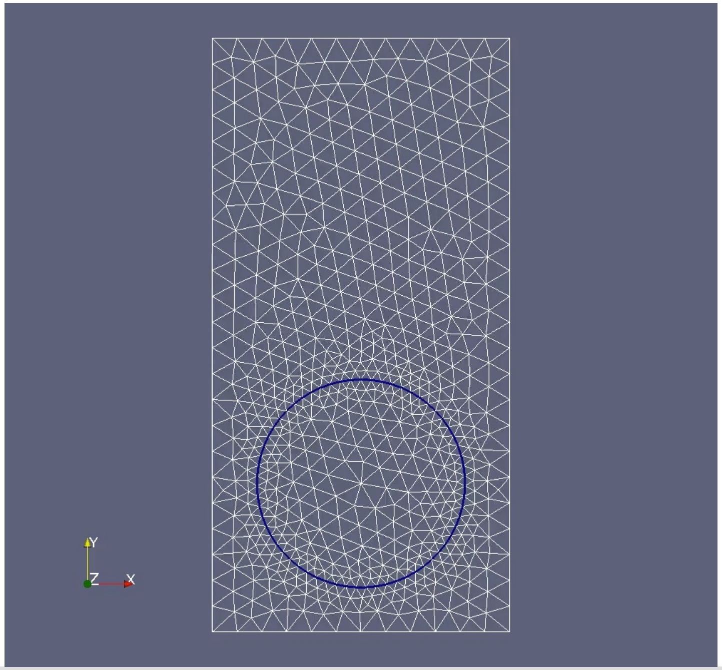
$$\min_{\varphi\in \mathrm{Emb}(M,\mathbb{D})} \frac{1}{2}\int_{\mathbb{D}} |y-\bar{y}|^2 \,\mathrm{d}x + \nu \int_{\varphi(M)} 1 \,\mathrm{d}s + \frac{\eta}{2}\int_{\varphi(M)} \left(g^M \circ \varphi^{-1}(s)\cdot \det D^\tau \varphi^{-1}(s) - f_\varphi(s)\right)^2 \mathrm{d}s$$

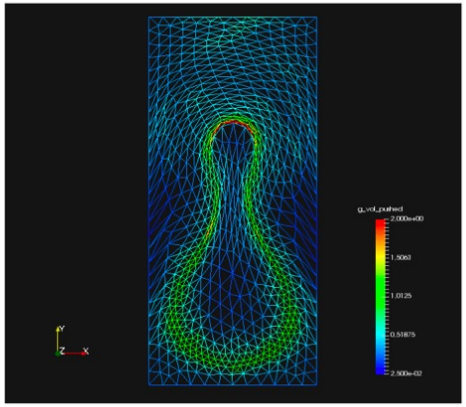
tang

$$\begin{aligned} \text{s.t.} \quad -\Delta y &= r_{\varphi(M)} \quad \text{in } \mathbb{D} \\ y &= 0 \quad \text{on } \partial\mathbb{D}. \end{aligned}$$

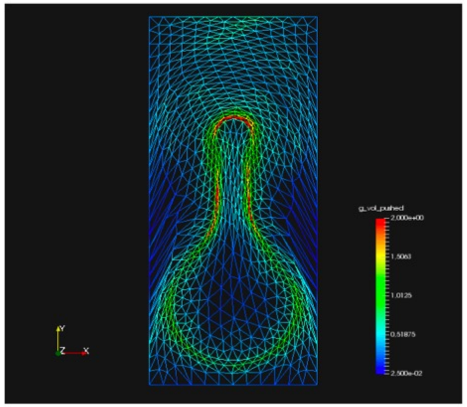




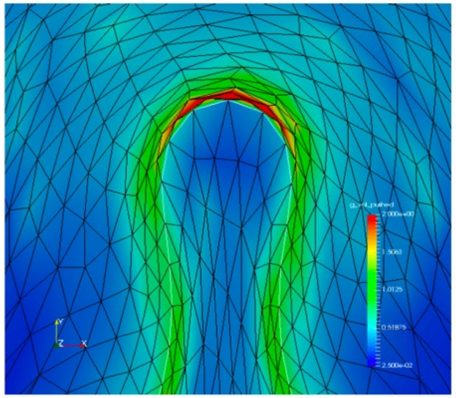




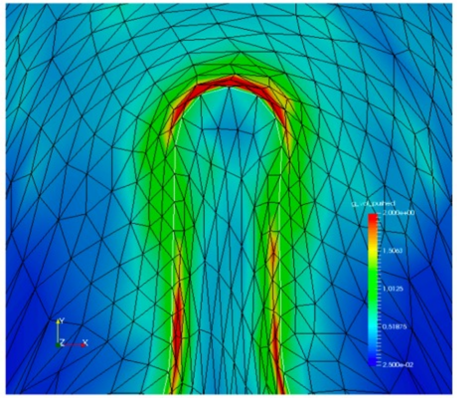
(a) Linear elasticity without regularization



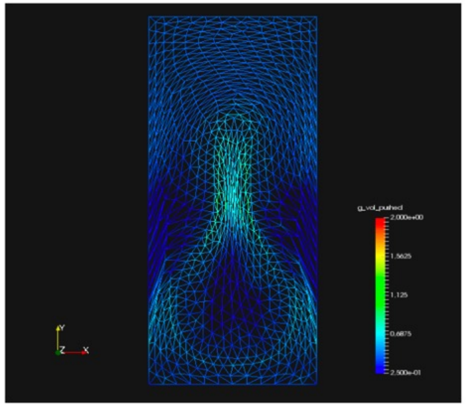
(b) Linear elasticity with tangential parameterization tracking



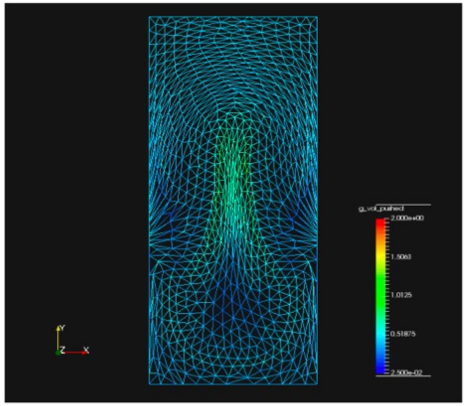
(a) Linear elasticity without regularization



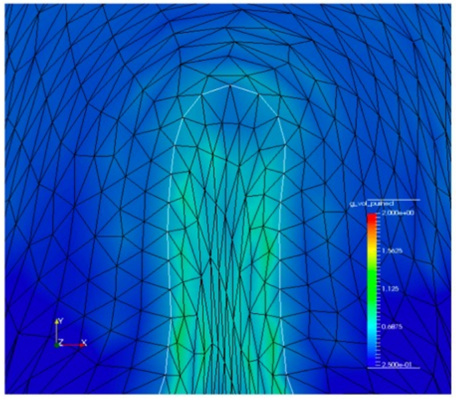
(b) Linear elasticity with tangential parameterization tracking



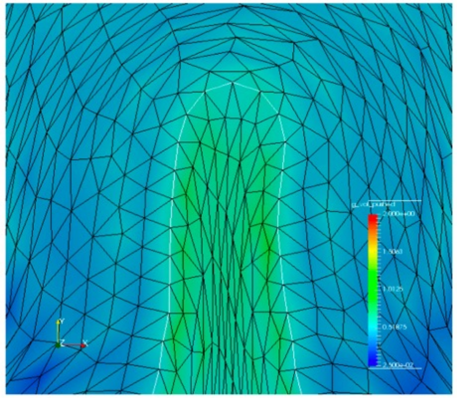
(c) Linear elasticity with tangential and volume parameterization tracking



(d) Linear elasticity with tangential and free outer boundary volume parameterization tracking



(c) Linear elasticity with tangential and volume parameterization tracking



(d) Linear elasticity with tangential and free outer boundary volume parameterization tracking

Further active topics

- Usage of Hadamard semiderivative for deriving shape derivatives for variational inequalities as in contact problems or Bingham fluids [Goldammer/Schulz/Welker arXiv:2208.03687]

- Shape calculus for nonlocal problems as in peridynamics or nonstandard diffusion [Schulz/Schuster/Vollmann arXiv:1909.08884]

Conclusion

- Shape optimization is vital in many applications.
- Focus on deformations rather than shapes gives a linear framework for algorithmic analysis, which acts as a linear "lifting" of the nonlinear shape space in minimal coordinates.
- Pre-shape calculus provides a framework for joint consideration of shape and mesh optimization.
- VI and nonlocal problems can be integrated within this framework