Modeling and Simulation of Lagrangian Mechanics through Automatic Differentiation and High-Index DAE Solving

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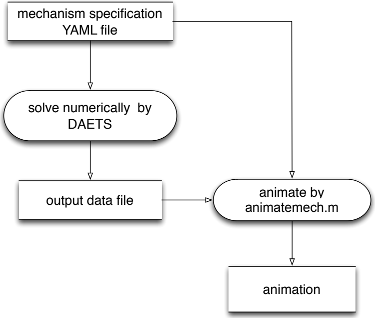
Introduction

- A multi-body system or mechanism is made of rigid parts, springs etc., connected by joints, moving under various forces.
- John & I realized our high-index differential-algebraic equation (DAE) solver DAETS is well suited to simulating mechanisms.
- On top of DAETS we built a Lagrangian facility.
  - Using automatic differentiation (AD), from a Lagrangian function, constraints,..., it constructs at runtime Euler-Lagrange equations, an index-3 DAE, which DAETS integrates.
- We re-invented—differently—the ~ 30 years established Natural Coordinates (NCs) approach.
- Mechanism facility (in progress)
  - reads text-file spec of mechanism, initial values (IVs) etc.
  - assembles Lagrangian, constraints, ...

System "Data, Lagrangian, Action"

- Data: reads text-file spec of mechanism, IVs etc.
- Creates Lagrangian; calls DAETS to solve and write output file
- Action: visualizes by our MATLAB code animatemech



Text-file is in YAML, a human-readable data serialization language.

Examples

"Mechanism1": ▸ Animate

Andrews Mechanism: ▸ Animate

Outline

DAETS solver

Lagrangian mechanics and using DAETS on it

Rigid body tracking

Example: the RSCR mechanism

Conclusion

References

More examples

DAETS overview

- DAETS— Differential Algebraic Equations by Taylor Series
- Solves DAE initial value problems, for state variables $x_j(t),\ j = 1:n$, of the general form

$$f_i(\,t, \text{ the } x_j \text{ and derivatives of them}\,) = 0, \quad i = 1:n.$$

- Can be fully implicit.
- d/dt can appear anywhere in the expressions for f_i
  e.g. one of the equations could be

$$\frac{\left((x_1' \sin t)'\right)^2}{1 + (x_2')^2} + t^2 \cos x_2 = 0.$$

DAETS's numerical method

- For some DAE with n equations $f_i = 0$ in terms of n variables $x_j(t)$ & their derivatives, define signature matrix $\mathbf{\Sigma} = (\sigma_{ij})$

$$\sigma_{ij} = \begin{cases} \text{highest order of derivative of } x_j \text{ occurring in } f_i \\ \text{or } -\infty \text{ if doesn't occur.} \end{cases}$$

- Structural analysis (SA) derives non-negative integer offsets:
  $c_1, \ldots, c_n$ of the equations; $d_1, \ldots, d_n$ of the variables.
- Form system Jacobian $\mathbf{J} = (J_{ij})$ where

$$J_{ij} = \frac{\partial f_i}{\partial x_j^{(d_j - c_i)}}, \qquad \text{or 0 where this makes no sense.}$$

($x^{(p)} = p$th time derivative of x.)

- Mathematically (for smooth f_i)

  method succeeds at $\mathbf{x}$ $\iff$ $\mathbf{J}$ nonsingular at $\mathbf{x}$

  where "succeeds" = "generates Taylor series of solution at $\mathbf{x}$".

DAETS solves high-index problems

- Index ν measures how difficult is to solve a DAE compared to an ODE (index 0).
- For traditional methods, $\nu \geq 3$ considered hard.
- Based on structural analysis of DAE + Taylor series.
    - Builds on AD package FADBAD++.
- In principle unaffected by index.
- Finding consistent initial point is done through IPOPT, "software library for large scale nonlinear optimization".

Lagrangian mechanics

- Lagrangian method is popular because it simplifies modeling.
  - Many ways to choose coordinates; all describe same motion.
- Lagrangian function

  $$\mathcal{L} = T - V$$

  $T =$ total kinetic energy, in terms of velocities and possibly positions.
  $V =$ total potential energy, caused by conservative (energy preserving) forces depending only on system position.
- Usually have constraints on motion—assume holonomic (velocity-independent) for simplicity—and external forces.

Lagrangian cont.

- Describe configuration at time t by vector $\mathbf{q} = (q_1, \ldots, q_{n_q})$ of n_q generalized position coordinates.
  $\dot{\mathbf{q}}$ is generalized velocities.
- Assumptions from previous slide imply

$$\mathcal{L} = T - V \qquad \text{with } T = T(\mathbf{q}, \dot{\mathbf{q}}),\ V = V(\mathbf{q}),$$

  plus n_c constraints on motion:

$$0 = C_i(t, \mathbf{q}), \qquad i = 1 : n_c$$

- Assuming the C_j are independent, the system has

$$\text{DOF} = n_q - n_c \quad \text{positional degrees of freedom.}$$

  Fix DOF q_j's and DOF $\dot{q}_j$'s to specify an initial value problem.

Lagrangian cont.

- Whatever coordinates chosen, variational "stationary action" principle gives (n_q+n_c) Euler–Lagrange equations of motion:

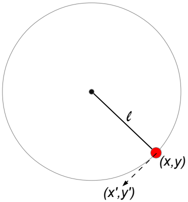
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} + \sum_{i=1}^{n_c} \lambda_i \frac{\partial C_i}{\partial q_j} = Q_j(t), \qquad j = 1:n_q \qquad (1)$$

$$C_i(t, \mathbf{q}) = 0, \qquad i = 1:n_c \qquad (2)$$

- λ_i are Lagrange multipliers for the constraints. $Q_j(t)$ are generalized external force components, if any (whose definition also involves $\partial/\partial q_j$).
- If $n_c = 0$ the system is of second kind, reducible to an ODE. If $n_c > 0$ it's of first kind and is an index 3 DAE.

Example: free motion of simple pendulum

Taking $\mathbf{q} = (x, y) =$ Cartesian coordinates of pendulum bob (of mass m) with y downward, gives



$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2), \qquad V = -mgy$$

$$\mathcal{L} = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy,$$

with one constraint that we write

$$0 = C = \tfrac{1}{2}(x^2 + y^2 - \ell^2)$$

Euler–Lagrange, on dividing through by m, give pendulum index-3 DAE

$$\begin{aligned}
0 = A &= \ddot{x} + \lambda x && \text{from } 0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial C}{\partial x} \\
0 = B &= \ddot{y} + \lambda y - g && \text{from } 0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} + \lambda \frac{\partial C}{\partial y} \\
0 = 2C &= x^2 + y^2 - \ell^2
\end{aligned}$$

Example: how Lagrangian facility works

- User encodes

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy \\ C &= x^2 + y^2 - \ell^2 \end{aligned} \tag{3}$$

- Lagrangian facility converts (3) to

$$\begin{aligned} 0 &= m\ddot{x} + 2x\lambda \\ 0 &= m\ddot{y} + 2y\lambda - mg \\ 0 &= x^2 + y^2 - \ell^2 \end{aligned}$$

which is of the form DAETS accepts.

Pendulum cont.

Alternatively, taking $\mathbf{q} = (\theta)$ = angle of pendulum from downward vertical, gives

$$T = \tfrac{1}{2}m(\ell\dot{\theta})^2, \qquad V = -mg\ell\cos\theta$$
$$\mathcal{L} = \tfrac{1}{2}m(\ell\dot{\theta})^2 + mg\ell\cos\theta$$

with no constraints. Then Euler–Lagrange lead to an ODE form

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta \qquad \text{from } 0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta}$$

which is equivalent to the DAE.

For one pendulum the angle model wins, but for $n > 1$ pendula (in a chain) the Cartesian model is **much** simpler . . .

Example: $n > 1$ pendula, in 3D Cartesians

- $\mathbf{r}_i = (x_i, y_i, z_i)$ position of ith bob (with z downward)
- Generalized coordinates
  $\mathbf{q} = (\mathbf{r}_1, \ldots, \mathbf{r}_n) = (x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$
- 1st kind formulation is

$$\left.\begin{aligned} \mathcal{L} &= \tfrac{1}{2} m \sum_{i=1}^{n} |\dot{\mathbf{r}}_i|^2 + mg \sum_{i=1}^{n} z_i \\ 0 = C_j &= |\mathbf{r}_j - \mathbf{r}_{j-1}|^2 - \ell^2, \qquad j = 1 : n \end{aligned}\right\} \tag{4}$$

  where $\mathbf{r}_0 = \mathbf{0}$, and $|\cdot|^2$ is the squared length of a 3-vector.
  - Constraints say the rods have length ℓ
  - $3n$ coordinate variables, n Lagrange multipliers
    Hence second-order DAE of size $4n$ and index 3
- DAETS with our Lagrangian facility solves (4) as written.

Example: The same, in ODE form

- Use spherical polar coordinates (θ_i, ϕ_i) for rod i
  - θ_i is rod's angle with downward vertical
  - ϕ_i is angle of rotation from the xz plane
- With $\mathbf{q} = (\theta_1, \phi_1, \ldots, \theta_n, \phi_n)$ we can get rid of the constraints.
- $2n$ coordinates, so $4n$ ODEs when reduced to first-order.
- Formulation is way more complex. E.g. KE is

$$T = \frac{1}{2} m\ell^2 \sum_{k=1}^{n} \left| \sum_{i=1}^{k} \begin{pmatrix} \cos\theta_i \,\dot{\theta}_i \cos\phi_i - \sin\theta_i \sin\phi_i \,\dot{\phi}_i \\ \cos\theta_i \,\dot{\theta}_i \sin\phi_i + \sin\theta_i \cos\phi_i \,\dot{\phi}_i \\ -\sin\phi_i \,\dot{\phi}_i \end{pmatrix} \right|^2$$

  and you still have the $\partial/\partial q_i, \partial/\partial \dot{q}_i$ stuff to do.
- It seems any other way to remove the constraints will use angles in some form.

Choices

Independent coordinates $n_c{=}0$	**Dependent coordinates** $n_c{>}0$
• Often most of the coordinates are angles.	NCs use Cartesian coordinates of points fixed on bodies.
• Eqns reduce to ODE, but messy.	Eqns are index-3 DAE but simple.
• "ODEs are nice to solve."	"DAEs are nasty to solve."
• Has become a big industry with some elegant maths.	NCs are a smaller industry, also with some nice maths.

DAETS is well suited to DAEs of the NCs kind (assume smooth).

- Based on structural analysis SA of the DAE.
  Theorem: SA "succeeds" on any DAE of this kind.
- Uses Taylor series of high order—typically 15–20.
- Infrastructure: IPOPT and FADBAD++.

Infrastructure benefits

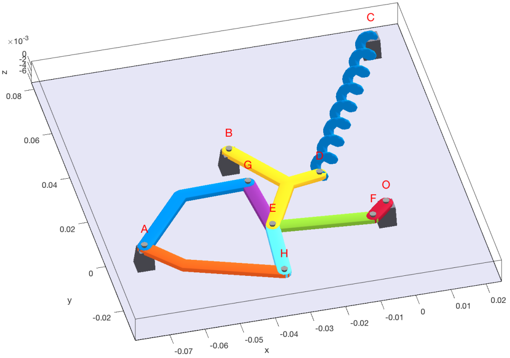
FADBAD++

- Inside DAETS forms Taylor series of solution, and system Jacobian.
- Outside DAETS, for a mechanism, forms Euler–Lagrange DAE from $\mathcal{L}$ and constraints by doing $\partial/\partial\mathbf{q}$, $\partial/\partial\,\dot{\mathbf{q}}$ and $\mathrm{d}/\mathrm{d}t$ at run time.

IPOPT

- Used by DAETS to find initial consistent point of DAE.
- For a mechanism this is initial position after fixing DOF IVs. IPOPT has proved very robust at finding this.
- Multiple solutions typical so need both fixed IVs & guesses.

E.g. in this Andrews Mechanism, G, H (independently) can lie either side of line AE.



How rigid motion is tracked

- Rigid motion of body $\mathcal{R}$ is described by "shift & rotate" map

$$\mathcal{R}_t = \mathsf{O}_t + \mathbf{Q}_t\mathcal{R}, \qquad \mathcal{R}_t \text{ is short for } \mathcal{R}(t)\text{, etc.} \qquad (*)$$

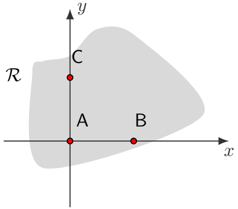
  $\mathcal{R}$: fixed in local frame
  $\mathcal{R}_t$: moving in world frame WF
  O_t: WF position of $\mathcal{R}$'s moving local origin
  $\mathbf{Q}_t$: rotation, whose columns $[\mathbf{i}_t, \mathbf{j}_t, \mathbf{k}_t]$ are $\mathcal{R}$'s moving orthonormal basis (ONB)

- (*) means that any point X and vector $\mathbf{u}$ fixed in $\mathcal{R}$ move in the WF according to

$$\mathsf{X}_t = \mathsf{O}_t + \mathbf{Q}_t\mathsf{X}, \qquad \mathbf{u}_t = \mathbf{Q}_t\mathbf{u}.$$

John–Ned tracking method

- NC methods express motion in terms of WF positions $A_t, B_t, \ldots$ of basic points or vectors (BPVs) $A, B, \ldots$ fixed on $\mathcal{R}$.
- In d dimensions, d BPVs "in general position" fix $\mathcal{R}$ uniquely.
- So in 3D we use 3 items A, B, C on $\mathcal{R}$.
  - A must be a point;
  - B, C can be either point or vector.



- "A = origin; B on +ve x axis; C in/toward upper xy plane."

- Adopt frame rule: $\mathcal{R}$'s local frame is unique coordinates s.t.

$$[\mathsf{A}, \mathsf{B}, \mathsf{C}] = \begin{bmatrix} 0 & \times & \times \\ 0 & 0 & \times \\ 0 & 0 & 0 \end{bmatrix}$$

and the upper triangular $\mathbf{R} = \begin{bmatrix} \times & \times \\ 0 & \times \end{bmatrix}$ has positive diagonal.

- From $\mathbf{R}$, and whether B, C is point or vector, we precompute constant matrix $\mathbf{U} = \begin{bmatrix} \times & \times \\ \times & \times \\ \times & \times \end{bmatrix}$ for each body such that

$$[\mathsf{A}_t, \mathsf{B}_t, \mathsf{C}_t]\ \mathbf{U} =: [\mathbf{i}_t, \mathbf{j}_t]$$

forms the first two vectors of $\mathcal{R}$'s moving ONB.

- Then cross product recovers the complete rotation matrix:

$$\mathbf{Q}_t = [\mathbf{i}_t, \mathbf{j}_t, \mathbf{i}_t \times \mathbf{j}_t].$$

To find $\mathcal{R}$'s KE, differentiate to get $\dot{\mathbf{Q}}_t$ and angular velocity vector $\boldsymbol{\omega}_t$, as function of $\dot{\mathsf{A}}_t, \dot{\mathsf{B}}_t, \dot{\mathsf{C}}_t$. (Omit, for time reasons.)

John–Ned versus classical NCs

- John–Ned NCs track $\mathcal{R}$'s motion by 3 BPVs, storing $3 \times 3 = 9$ scalars/body, but using nonlinear operation $\mathbf{i}_t \times \mathbf{j}_t$.
- Classical NCs track by BPVs: $4 \times 3 = 12$ scalars/body, but avoiding nonlinear operation.
  (In both cases fewer on average due to sharing BPVs between bodies.)
- Difference: nonlinearity gives us a varying mass matrix $\mathbf{M}(\mathbf{q})$, while classical NCs produce a constant $\mathbf{M}$.

  Larger but constant versus smaller but varying.
  - We have more compact models.
  - DAETS handles without difficulty time varying mass matrices.

Assembly example: the RSCR

Revolute-Spherical-Cylindrical-Revolute

A well studied 3D mechanism with 1 DOF.

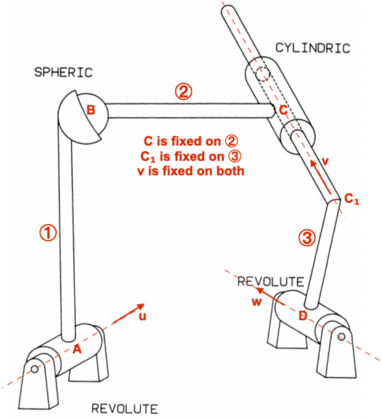
Basic points/vectors of parts:

① AB$\mathbf{u}$

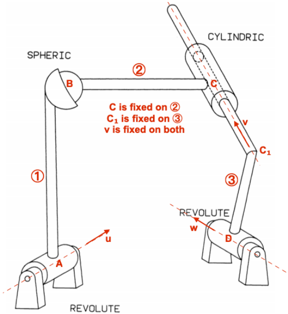
② CB$\mathbf{v}$

③ $\text{DC}_1\mathbf{w}\mathbf{v}$

The cylindric joint is made by ② and ③ sharing $\mathbf{v}$.



Without loss, assume
– angles at A, C, C_1, D are 90^o
and in WF,
– D=origin, A on –ve x axis;
– $\mathbf{w}$ in xy plane at angle δ to x axis;
– $\mathbf{u}$ at angle $\langle \alpha, \beta \rangle$ in spherical polars.
and in ③,
– $\mathbf{v}$ makes angle γ with $\mathbf{w}$.



(Design params: 3 lengths, 4 angles.)
Take $\mathbf{q} = (\mathrm{C}_1, \mathrm{B}, \mu)$, where μ is a scalar used in cylindric joint.
Total 7 scalars.

RSCR kinematics & degrees of freedom

By assignments we'll define vector $\mathbf{v}$ in ③ as function of C_1 only, then share it with ② (underscore means fixed in WF):

From q	Assign	Equate	DOF
Define ③'s frame vectors, hence $\mathbf{v}$, and ③'s rigidity			
C_1	$\mathbf{i}_3 := \mathsf{C}_1/L_3$	$0 = \mathsf{C}_1^2 - L_3^2$	$+3\ -2$
	$\mathbf{k}_3 := \mathbf{i}_3 \times \underline{\mathbf{w}}$	$0 = \mathsf{C}_1 \cdot \underline{\mathbf{w}}$	
	$\mathbf{v} := \cos(\gamma)\underline{\mathbf{w}} + \sin(\gamma)\mathbf{k}_3$		
Define cylindric joint, and ②'s rigidity			
B, μ	$\mathsf{C} := \mathsf{C}_1 + \mu\mathbf{v}$	$0 = (\mathsf{B} - \mathsf{C})^2 - L_2^2$	$+3{+}1\ -2$
		$0 = (\mathsf{B} - \mathsf{C}) \cdot \mathbf{v}$	
Define ①'s rigidity			
		$0 = (\mathsf{B} - \underline{\mathsf{A}})^2 - L_1^2$	-2
		$0 = (\mathsf{B} - \underline{\mathsf{A}}) \cdot \underline{\mathbf{u}}$	
		Net DOF:	$1 = +7\ -6$

RSCR, cont.

- To simplify dynamics let the parts be thin uniform rigid rods AB, BC, C_1D of masses m_1, m_2, m_3 (let joints be massless).

  Such a rod of mass m, moving with ends at $Y(t), Z(t)$ in WF, has KE $\frac{m}{6}(|\dot{Y}|^2 + \dot{Y}\cdot\dot{Z} + |\dot{Z}|^2)$. Then

$$\begin{aligned} T &= T_1 + T_2 + T_3 \quad (\text{Note A, D are fixed so } \dot{A} = \dot{D} = \mathbf{0}) \\ &= \tfrac{m_1}{6}(|\dot{B}|^2) + \tfrac{m_2}{6}(|\dot{B}|^2 + \dot{B}\cdot\dot{C} + |\dot{C}|^2) + \tfrac{m_3}{6}(|\dot{C}_1|^2) \\ &= \tfrac{1}{2}\,\dot{\mathbf{q}}^T \big(\underbrace{\mathbf{M}_1}_{\text{const}} + \mathbf{M}_2(\mathbf{q}) + \underbrace{\mathbf{M}_3}_{\text{const}}\big)\,\dot{\mathbf{q}}\,. \end{aligned}$$

- Dependence of $\mathbf{M}_2$ on $\mathbf{q}$ is because

$$\dot{C} = \dot{C}_1 + \mu\dot{\mathbf{v}} + \dot{\mu}\mathbf{v}$$

  is nonlinear in $\mathbf{q}$.

Compare sizes of system

- This shows length n_q of $\mathbf{q}$, and number n_c of constraints, in Lagrangian formulation of two examples, by classical "Jalon–Bayo" Natural Coordinates (JBNCs), and the Nedialkov–Pryce kind (NPNCs).
- Resulting DAE size is $n_q + n_c$, but $2n_q + n_c$ (in parentheses) in 1st order form for solvers such as DASSL.

	1 DOF RSCR		6 DOF robot	
	JBNCs	NPNCs	JBNCs	NPNCs
#coords n_q	13	7	30	18
#constraints n_c	12	6	24	12
DAE size	25 (38)	13	54 (84)	30

- Our savings are mainly from more economical frames, and use of assignments instead of equations where possible.

Conclusion

- Modelling: write Lagrangian, constraints, etc. in Cartesian coordinates.
  Much simpler than using angle coordinates and converting to an ODE.
- Since high-index DAEs are now as easy to solve as ODEs, a Lagrangian formulation needn't avoid constraints.
- So rigid-body mechanical systems can be modeled in Cartesian coordinates, which is simpler.
- This makes the concept so easy that Lagrangian stuff can be taught at undergraduate level.

- Teaching. John & I are late to this party, but:
  - Classical NC-ers have argued for years: Cartesian coordinates make MBS ideas so easy that Lagrangian stuff can be taught at undergraduate level. We agree.
  - We can add: with DAETS the DAEs are now as easy as ODEs.
- Our infrastructure should be good at important tasks such as:
  - Finding stationary equilibrium configuration.
  - Inverse dynamics: torques etc. that give a desired motion.
  - Kinematic design, e.g. make RSCR move optimally near a desired curve.

References

- JP, NN. (2020) Multibody Dynamics in Natural Coordinates through Automatic Differentiation and High-Index DAE Solving
  Acta Cybernetica 24, 315–341
- JP, NN, G. Tan, X. Li (2018) How AD can help solve differential-algebraic equations
  Optimization Methods and Software, 33:4-6, 729-749
- B. Derakhshan. Multibody Dynamics Problems in Natural Coordinates: Theory, Implementation and Simulation
  M.Sc. thesis, Computational Science and Engineering, McMaster University, 2022

- YouTube Multi-body Lagrangian Simulations
    - Outer planets
    - Gravitating masses in 2D
    - Rods, plate, springs, particles, collinears
    - RSCR Chebyshev linkage version

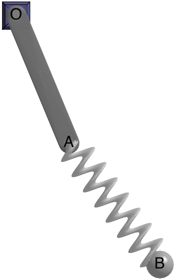
    ⋮

Example: Mechanism0

- Rod $\mathbf{OA}$ pivots at fixed point O.
- Spring $\mathbf{AB}$ attached at $\mathbf{A}$.
- Particle attached at B.
- System moves under gravity.
- Animation

Mechanism0: specification in YAML

```
Title:  Mechanism0
Dimension: 2
PhysicalParams:
  { L: 1,  M: 1,  k:  500,  l :  1,  mu: .5,  m: 2 }
PartData:
  Fixed: { O: [0,  0]  }
  Rigids:
    OA:
      Geom: [[L, 0]]  # local  frame geometry
      #      centroid ,  mass, moment of inertia
      Dyna: [[L/2, 0],  M, M*L**2/12]
  Springs:
    AB: [k,  l ,  mu] # stiffness ,  length ,  mass
  Particles :
    B:  m # mass
AppliedForces:
  Gravity:  #turns it  on with default  value
```

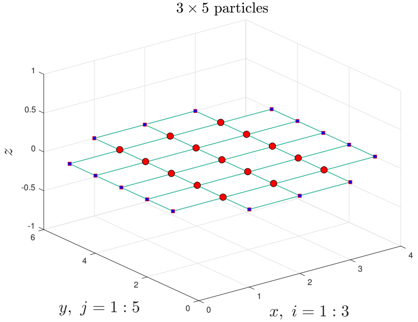


Mechanism0: IVs; solver & animation settings

```
ProblemData: # integration interval, etc.
  t0: 0
  tend: 20
  positions : { A: [L, 0], B: [(L+1), 0] } # guesses for A, B
  fixedpositions : { Ax, B } # don't change values for A.x, B
  velocities : { Ay: 10, B: [0, 0] }
SolverParams: # to guide DAETS
  Integration :
    tol : 1e-12
    order: 20
Animation: # to guide animation
  Scene:
    view: [0, 90] # camera azimuth, elevation
```

Larger example: Particle-spring system

- Rectangular grid of $m \times n$ particles connected by damped springs.
- A test for cloth simulation in movies.



- Particle (i, j) is attached to

$$(i \pm 1, j) \text{ and } (i, j \pm 1) \text{ for } i = 1:m,\ j = 1:n$$

- Index $i = 0$ or $m + 1$, resp. $j = 0$ or $n + 1$, means a fixed position.

Particle-spring cont

- Each particle (i, j)
  - coordinates $\boldsymbol{r}_{ij} = (x_{ij}, y_{ij}, z_{ij})$ full 3D motion
  - mass M
- Each spring
  - stiffness k
  - length at rest l
  - damping $k_d\times$ stretch-rate (except the boundary ones)
- Spacing Δx and Δy between particles in x and y directions.
- Initially all particles at rest in xy plane, we push the middle particle upwards.
- Animation 90×90 particles, 24,300 second-order ODEs.

Particle-spring cont

- Energy contributions

	KE	PE
Particle (i,j)	$\frac{1}{2}M\lvert\dot{\boldsymbol{r}}_{ij}\rvert^2$	Mgz_{ij}
Spring (i,j)–$(i,j+1)$		$\frac{1}{2}(\lvert\boldsymbol{r}_{i,j+1}-\boldsymbol{r}_{ij}\rvert-l)^2$

  Add up over whole mesh to get T, V and then $\mathcal{L}=T-V$.

- Spring (i,j)–$(i,j+1)$ dissipation

$$\frac{1}{2}k_d\lvert\dot{\boldsymbol{r}}_{i,j+1}-\dot{\boldsymbol{r}}_{ij}\rvert^2$$

  Add up to obtain Rayleigh dissipation function R.

- Lagrangian facility takes $\mathcal{L}$ and R and produces 2nd-order ODE of size $3mn$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{r}}_{ij}}-\frac{\partial\mathcal{L}}{\partial\boldsymbol{r}_{ij}}+\frac{\partial R}{\partial\dot{\boldsymbol{r}}_{ij}}=0, \qquad i=1:m,\ j=1:n$$

  which DAETS integrates.